INDEX

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Significance of Diff Geom in the
Science of Cartography
History of Cartography
Imp. of Cartography in 15 - 19 cent.
Finnish Carto.
Loxodromes etc.

The Spectral Theorem (Purely matrix theoretic approach)

Prop: The eigen values of a real symmetric matrix are real numbers

proof: Let A be a real symmetric matrix and λ be a (possibly complex) eigen value of A
Let v be its eigen vector in $\mathbb{C}^n$.

$A v = \lambda v$. Taking complex conjugate

$A \bar{v} = \bar{\lambda} \bar{v}$

$\therefore \quad v^T A \bar{v} = \bar{\lambda} v^T \bar{v} = \bar{\lambda} \|v\|^2$

$\therefore \quad (Av)^T \bar{v} = \bar{\lambda} \|v\|^2$

$\therefore \quad \lambda v^T \bar{v} = \bar{\lambda} \|v\|^2$ or $\lambda \|v\|^2 = \bar{\lambda} \|v\|^2$

Since $\|v\| \neq 0$, $\lambda = \bar{\lambda}$

Recall that a complex matrix A is Hermitian or self adjoint if $A = A^*$ (the conjugate transpose of A)

Ex: Show that the eigen values of a Hermitian matrix are real.

Prop: Suppose λ, μ are distinct eigen values of a Hermitian matrix and v, w are the corresp. eigen vectors then

$\langle v, w \rangle = 0$ where

$\langle v, w \rangle = \sum v_i \bar{w}_i$

The corresponding result is true for real symm. matrices except that in this case $v, w \in \mathbb{R}^n$

proof: $A v = \lambda v$

$A w = \mu w$

$\therefore \quad \bar{w}^T A v = \lambda \langle w, v \rangle$

$\therefore \quad (\overline{A w})^T v = \lambda \langle w, v \rangle$

∴ $(\overline{\mu w})^T v = \lambda \langle w, v \rangle$

∴ $\mu \langle w, v \rangle = \lambda \langle w, v \rangle$

(since $\mu = \bar{\mu}$)

∴ $\langle w, v \rangle = 0$ since $\lambda \neq \mu$.

Thm (Spectral theorem):

(1) An $n \times n$ real symm matrix has an orthonormal basis of eigen vectors in $\mathbb{R}^n$

Equivalently ∃ P orthogonal s.t.

$$P^T A P = \text{diagonal matrix}$$

(2) An $(n \times n)$ Hermitian matrix has an orthonormal basis of eigen vectors in $\mathbb{C}^n$

Equivalently ∃ a unitary matrix U s.t.

$$U^* A U = \text{diag matrix}$$

(Note: U unitary simply means $U^* U = I_n = U U^*$)

Each of these statements may be described as: A real symm. or hermitian matrix is unitarily diagonalizable.

Proof: The proofs are similar and we only do the real case.

Let λ_1 be an eigen value of A with eigen-vector v_1 of unit length.

Choose vectors $u_2, \dots, u_n$ such that $\{v_1, u_2, \dots, u_n\}$ is an orthonormal basis of $\mathbb{R}^n$ and put

$$P_1 = [v_1, u_2, \dots, u_n]$$ which is an orthogonal matrix.

$$AP_1 = [Av_1, Au_2, \dots, Au_n]$$

(Well, to see this apply both sides to $\hat{e}_k$ and recall that for any matrix C, $C\hat{e}_k$ is the kth column of C).

$$\therefore AP_1 = [\lambda_1 v_1, Au_2, \dots, Au_n]$$

We now wish to calculate $P_1^T A P_1$

The jth row of $P_1^T = u_j^T$ $(j \geq 2)$
$= v_1^T$ $(j = 1)$

$$v_1^T A P_1 = [\lambda_1, v_1^T A u_2, \dots, v_1^T A u_n] = [\lambda_1, \lambda_1 \langle v_1, u_2 \rangle, \dots, \lambda_1 \langle v_1, u_n \rangle] = [\lambda_1, 0, \dots, 0].$$

For $j \geq 2$

$$u_j^T A P_1 = [0, u_j^T A u_2, \dots, u_j^T A u_n] \quad (*)$$

Thus $$P_1^T A P_1 = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & B & \vdots \\ 0 & \cdots & & 0 \end{bmatrix} = \tilde{A} \text{ say}$$

where B is an $(n-1) \times (n-1)$ matrix

$\cancel{\tilde{A}\tilde{A}^T = (P^TAP)(P^TAP)^T = P^T}$

Since $(P^TAP)^T = P^TAP$ we see that $\tilde{A}$ and hence B is real symmetric.

By induction hypothesis B has an orthonormal basis $w_2, \dots, w_n$ of eigen vectors in $\mathbb{R}^{n-1}$ with eigen values $\lambda_2, \dots, \lambda_n$ say; $\|w_2\| = \dots = \|w_n\| = 1$.

Put $v_j = P_1 \begin{bmatrix} 0 \\ w_j \end{bmatrix}$ $\quad j = 2, 3, \dots, n$

Then $Av_j = AP_1 \begin{bmatrix} 0 \\ w_j \end{bmatrix}$

To compute this (without relying on block multiplication of matrices. We are proving everything from scratch)

$$v_1^T A v_j = v_1^T A P_1 \begin{bmatrix} 0 \\ w_j \end{bmatrix}$$

$$= \lambda_1 v_1^T P_1 \begin{bmatrix} 0 \\ w_j \end{bmatrix}$$

$$= \lambda_1 [1, 0, \dots, 0] \begin{bmatrix} 0 \\ w_j \end{bmatrix} = 0$$

and for $k \geq 2$

$$u_k^T A v_j = u_k^T A P_1 \begin{bmatrix} 0 \\ w_j \end{bmatrix}$$

$$= [0, R_k(B)] \begin{bmatrix} 0 \\ w_j \end{bmatrix}$$

using (*), where $R_k(B)$ denotes the k-th row of B

$\therefore$ $u_k^T A v_j = \lambda_j R_k(w_j)$ (kth row of w_j)

$\therefore$ $P_1^T A v_j = \lambda_j \begin{bmatrix} 0 \\ w_j \end{bmatrix}$ or

$A v_j = \lambda_j v_j$ $(j = 2, 3, \dots, n)$

Thus $v_1, \dots, v_n$ are eigen vectors of A.

$$\langle v_j, v_\ell \rangle = \left(P_1 \begin{bmatrix} 0 \\ w_\ell \end{bmatrix}\right)^T \left(P_1 \begin{bmatrix} 0 \\ w_j \end{bmatrix}\right)$$

$$= [0, w_\ell^T]^T P^T P_1 \begin{bmatrix} 0 \\ w_j \end{bmatrix}$$

$$= \langle w_\ell, w_j \rangle = \delta_{j\ell} \quad (j, \ell \geq 2)$$

If $j = 1$ and $\ell \geq 2$

$$\langle v_1, v_\ell \rangle = \left(P \begin{bmatrix} 0 \\ w_\ell \end{bmatrix}\right)^T v_1$$

$$= [0, w_\ell^T] P^T v_1$$

$$= [0, w_\ell^T] \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

Thus the vectors $v_1, \dots, v_n$ are orthonormal.
The proof is complete.

Theorem: (1) If A, B are commuting real symm matrices, they have a common eigen vector in $\mathbb{R}^n$

(2) If A, B are commuting Hermitian matrices they have a common eigen-vector in $\mathbb{C}^n$

Proof: Let λ be an eigen value of A and $v_1, \dots, v_p$ be the complete set of eigen vectors of A corresponding to the eigen value λ.

$$A(Bv_j) = B(Av_j) = \lambda Bv_j$$

$$\therefore \quad Bv_j = \sum_{k=1}^{p} c_{jk} v_k$$

$$c_{jk} = v_k^T B v_j \quad \text{which is a } 1\times 1 \text{ matrix}$$

$$= (v_j^T B^T v_k)^T$$

$$= v_j^T B v_k = c_{kj}$$

The matrix C is a $p \times p$ real symm matrix and so has eigen value μ with eigen vector $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$ say. So $\alpha_1, \dots, \alpha_p$ are not all zero and $v_1, \dots, v_p$ are lin. indep $\Rightarrow u = \alpha_1 v_1 + \dots + \alpha_p v_p \neq 0$.

Claim: u is the common eigen-vector for A and B

It is obviously an eigen vector for A.

$$Bu = \sum_j \alpha_j Bv_j$$
$$= \sum_{j,k} \alpha_j c_{jk} v_k$$
$$= \sum_k \left(\sum c_{kj} \alpha_j \right) v_k$$
$$= \sum_k \mu \alpha_k v_k = \mu u$$

The proof is complete.

Theorem: (1) If A, B are commuting real symmetric matrices they have a common orthonormal basis of eigen vectors in $\mathbb{R}^n$ i.e $\exists$ P orthogonal such that $P^T A P$ and $P^T B P$ are both diagonal.

(2) If A, B are commuting Hermitian matrices, they have a common basis of eigen-vectors in $\mathbb{C}^n$. That is, $\exists$ a unitary matrix U such that $U^* A U$ and $U^* B U$ are both diagonal.

Proofs: First choose a common eigen vector v_1 for both A and B: $\|v_1\| = 1$

Choose an orthonormal basis $v_1, u_2, \ldots, u_n$ and

$$P_1 = [v_1, u_2, \ldots, u_n]$$

As in the proof of spectral theorem

$$P_1^T A P_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & E \end{bmatrix}$$

$$P_1^T B P_1 = \begin{bmatrix} \mu_1 & 0 \\ 0 & F \end{bmatrix}$$

where E, F are real symm of size $(n-1) \times (n-1)$.

$$(P_1^T A P_1)(P_1^T B P_1) = (P_1^T B P_1)(P_1^T A P_1)$$
$$\Rightarrow \quad EF = FE$$

So we can induct on the size of A and B.
Thus we may assume inductively that E, F have a common basis of eigen vectors $w_2, \ldots, w_n$ in $\mathbb{R}^{n-1}$ s.t $\langle w_i, w_j \rangle = \delta_{ij}$

But as in the spectral theorem if we set $v_j = P_1 \begin{bmatrix} 0 \\ w_j \end{bmatrix}$ then both

$$Av_j = \lambda_j v_j \; ; \quad Bv_j = \mu_j v_j \quad \text{would hold}$$

(λ_j = eigen value of E with eigen-vector w_j
μ_j = eigen value of F with eigen-vector w_j)
thereby providing us with the common basis of eigen vectors.

Remark: Thus commuting real symm (or Hermitian) matrices can be simultaneously unitarily diagonalized.

Normal Matrices: A square matrix A with complex entries is said to be normal if
$$AA^* = A^*A$$
Thus Hermitian and Unitary matrices are normal.

Prop. Given a normal matrix A, $\exists\,!$ Hermitian matrices B and C such that
(i) $A = B + iC$
(ii) $BC = CB$.

<u>Proof:</u> Take $B = \frac{1}{2}(A + A^*)$

$$C = \frac{1}{2i}(A - A^*)$$

Then B and C are Hermitian and $BC = CB$ since $AA^* = A^*A$

If $A = P + iQ$ where P, Q are hermitian and commuting,

$A^* = P - iQ$ and we see that $P = B$ and $Q = C$ proving uniqueness.

Since B, C can be simultaneously unitarily diagonalized we get

<u>Theorem:</u> Every normal matrix has an orthonormal basis of eigen vectors i.e can be unitarily diagonalized

Conversely, if an $(n \times n)$ matrix is unitarily diagonalizable it is normal.

<u>proof:</u> Only the last part needs a proof

Let A be unitarily diagonalizable

$$\therefore \quad U^*AU = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = D$$

$$\therefore \quad U^*A^*U = \begin{bmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{bmatrix} = \bar{D}$$

Since D and $\bar{D}$ commute

$$(U^*AU)(U^*A^*U) = (U^*A^*U)(U^*AU)$$

$\therefore \quad AA^* = A^*A$. The proof is complete.

II - Theory of plane Curves

A parametrized curve in $\mathbb{R}^n$ is a mapping
$\gamma: I \to \mathbb{R}^n$, where I is an open interval in $\mathbb{R}$, called the parameter interval, and γ has derivatives of all orders.

A parametrized curve γ is said to be regular if $\gamma'(t) \neq 0$ for any $t \in I$.

Note that a curve is not just a point set.

Thus $\gamma_1: \mathbb{R} \to \mathbb{R}^2$ and $\gamma_2: \mathbb{R} \to \mathbb{R}^2$ given by

$$\gamma_1(t) = (\cos t, \sin t)$$
$$\gamma_2(t) = (\cos 2t, \sin 2t)$$

are two distinct curves through they both describe the unit circle $\{(x,y) \in \mathbb{R}^2 / x^2+y^2=1\}$ as a point-set. The image $\{\gamma(t) / t \in I\}$ is called the trace of the curve.

<u>Ex</u>: Show that if γ_1, γ_2 are two curves with the same parameter interval I then

$$\frac{d}{dt} \langle \gamma_1, \gamma_2 \rangle = \langle \gamma_1, \dot{\gamma}_2 \rangle + \langle \dot{\gamma}_1, \gamma_2 \rangle$$

We shall use interchangably the notations

$$\gamma'(t) = \dot{\gamma}(t) = \frac{d}{dt}\gamma(t).$$

$\| x \|$ will always denote the Euclidean norm $(x_1^2 + \dots + x_n^2)^{1/2}$.

<u>Def</u>: Given a parametrized curve $\gamma: I \to \mathbb{R}^n$ the vector $\dot{\gamma}(t)$ is the tangent vector to the curve at the point $\gamma(t)$ on the curve.

Def: $\|\dot{\gamma}(t)\|$ is called the speed of the curve.

Ex: Trace the curve $\gamma: \mathbb{R} \to \mathbb{R}^2$ given by
$$\gamma(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$$ and determine its speed

(2) Show that if $\hat{a}, \hat{b}$ are linearly independent unit vectors in the plane then
$\gamma(t) = \hat{a}\cos t + \hat{b}\sin t$, $t \in \mathbb{R}$
describes an ellipse. Determine its major and minor axes.

(3) Show that the curve $\gamma: \mathbb{R} \to \mathbb{R}^3$
$\gamma(t) = (a\cos t, a\sin t, bt)$ lies on a cylinder and find the tangent vector at any point on it. Sketch the curve.

(4) Use Exercise (2) to write out a parametrization for the circle of intersection of $x^2+y^2+z^2=1$ and the plane $x+y+z=0$
Do the same for the circle of intersection of
$x^2+y^2+z^2=1$ and $x+y+z = \frac{1}{\sqrt{3}}$.

(5) Consider a quadratic form $Q(\vec{x})$ given by $x^T A x$ where A is an $n \times n$ real symmetric matrix.
Discuss conditions under which the quadric
$$x^T Q x = 1$$
~~has~~ contains ~~straight~~ lines. ~~by attempting~~
If so, how many such lines pass through

a given point $\vec{x}_0$ on the quadric?

Sol: If such lines exist they are given by $\vec{x}_0 + \hat{u}t$ and so
$$x_0^T A u = 0$$
$$u^T A u = 0 .$$
Thus the direction of the line is the intersection of a hyperplane and a cone through the origin
(Look at the cases $n=3$, $n>3$)

(6) Sketch the curve $\gamma: \mathbb{R} \to \mathbb{R}^2$ given by
$\gamma(t) = (a\cos^3 t, b\sin^3 t)$.
Discuss the regularity of the curve.

(7) Show that the curve $\gamma: \mathbb{R} \to \mathbb{R}^3$ given by
$\gamma(t) = (a_0 + a_1 t + a_2 t^2;\ b_0 + b_1 t + b_2 t^2;\ c_0 + c_1 t + c_2 t^2)$
lies on a plane. What kind of curve is it?

(8) A pendulum is suspended at the origin and is set in motion ~~by~~ from its mean position ~~by~~ with a velocity v. The pendulum rises and reaches its ~~an~~ unstable equilibrium as $t \to +\infty$
Find the trajectory of the bob as a parametrized curve.

Arc Length

Let $\gamma : I \to \mathbb{R}^n$ be a parametrized curve and $c, d \in I$ with $c < d$.
The arc length of the piece of the curve $\gamma|_{[c,d]}$ is given by

$$L_\gamma[c,d] = \int_c^d \|\gamma'(t)\|\, dt \quad \text{denoted by } L_\gamma[c,d]$$

Exercise: Explain intuitively why this definition makes sense. Arrive at this formula by heuristic considerations such as approximating the curve by polygons ~~and~~ obtaining thereby Riemann sums for the integral.

In particular if we fix $a_0 \in I$ and take an arbitrary point $t \in I$ ($s < a_0$ not excluded) then

$$L_\gamma[a_0, t] = \int_{a_0}^{t} \|\gamma'(\lambda)\|\, d\lambda$$

Note that if $t < a_0$ then $L_\gamma[a_0, t]$ would be negative and if $t = a_0$ it is zero.
The function $L_\gamma[a_0, t] = s(t)$ for short is called the arc length function defined on I.
Thus $s(t)$ is the oriented distance measured along the curve from a chosen reference point $\gamma(a_0)$ on the curve.

Reparametrization: Let $\gamma : I \to \mathbb{R}^n$ be a parametrized curve which is regular and $\phi : J \to I$ be a smooth map with a smooth inverse. Then the curve $\gamma \circ \phi : J \to \mathbb{R}^n$, is called a reparametrization of γ.

The reparametrization is said to be proper if or orientation preserving if $\phi' > 0$ on J.

Theorem: The arc length is invariant under proper reparametrization. Thus if ϕ is as above then

$$L_{\gamma\circ\phi}[c,d] = L_\gamma[\phi(c), \phi(d)]$$

Proof:
$$L_{\gamma\circ\phi}[c,d] = \int_c^d \|(\gamma\circ\phi)'(\lambda)\|\, d\lambda$$
$$= \int_c^d \|\gamma'(\phi(\lambda))\|\, \phi'(\lambda)\, d\lambda$$
$$= \int_{\phi(c)}^{\phi(d)} \|\gamma'(u)\|\, du$$
$$= L_\gamma[\phi(c), \phi(d)]$$

Proof is complete.

Fixing $c = \alpha$ and varying d one can think of using the arc length function to reparametrize a regular curve.

Note that if $\gamma : I \to \mathbb{R}^n$ is a regular curve then $\frac{ds}{dt} = \|\gamma'(t)\| > 0$. Fixing $t = a_0 \in I$ we get a function $t \overset{\phi}{\mapsto} \int_{a_0}^{t} \|\gamma'(\lambda)\|\, d\lambda$ denoted by the variable s i.e $s = \phi(t)$

$$\phi : [a_0, t] \longrightarrow [0, L_\gamma[a_0,t]] \quad (t > a_0)$$

The reparametrization of γ via arc length is then the parametrized curve

$$\sigma = \gamma \circ \phi^{-1}$$

Now $\frac{d\sigma}{ds} = \gamma'(\phi^{-1}(s)) \frac{d}{ds}\phi^{-1}(s)$

$$= \frac{\gamma'(\phi^{-1}(s))}{\phi'(t)} \quad ; \quad s = \phi(t)$$

$$= \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

Thus $\|\sigma'(s)\| = 1$ and the arc length reparametrization produces a unit speed curve.

In what follows (examples and exercises apart) a curve will always mean a regular parametrized ~~unit speed curves~~ curve with unit speed.

Ex: What happens if we reparametrize a unit speed curve by arc length?

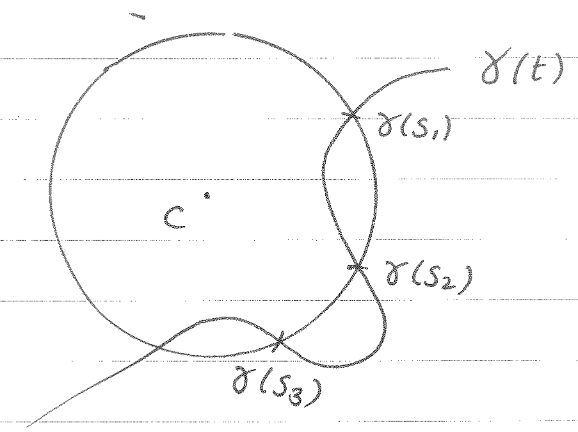
(9) Exercises: Find the arc length function for the conical helix $\gamma : I \to \mathbb{R}^3$ given by
$$\gamma(t) = (e^t \cos t,\ e^t \sin t,\ e^t)$$

(10) Exercise: On the sphere $x^2+y^2+z^2=1$ sketch the meridians and determine all curves on the sphere that cut each meridian at an angle of $45°$.
Such curves are called loxodromes.

(11) If all normals to a curve pass through the origin show that the curve is a circle.
If all tangent lines to a regular curve pass through a fixed point, the curve is a line.
Discuss what happens when the regularity assumption is dropped?

The Osculating Circle and center of Curvature.

Let $\gamma : I \to \mathbb{R}^2$ be a regular parametrized plane curve. parametrized by arc length and let $s_1, s_2, s_3 \in I$. be three distinct values of the parameter.



Let $C(s_1, s_2, s_3)$ be the center of the circle through $\gamma(s_1)$, $\gamma(s_2)$ and $\gamma(s_3)$ (if it exists).

The function

$$\rho(s) = (\gamma(s) - C(s_1, s_2, s_3)) \cdot (\gamma(s) - C(s_1, s_2, s_3)) \qquad (1)$$

vanishes at s_1, s_2, s_3 so that by Rolle's theorem ρ' vanishes on (s_1, s_2) and (s_2, s_3) say at u_1 and u_2. Again, ρ'' vanishes on (u_1, u_2); say $\rho''(v) = 0$.

Now

$$\rho'(s) = 2\gamma'(s) \cdot (\gamma(s) - C(s_1, s_2, s_3)) \qquad (2)$$

$$\rho''(s) = 2 + 2\gamma''(s) \cdot (\gamma(s) - C(s_1, s_2, s_3)) \qquad (3)$$

$$\therefore \gamma''(v) \cdot (\gamma(v) - C(s_1, s_2, s_3)) = -1$$

$$\gamma'(u_j) \cdot (\gamma(u_j) - C(s_1, s_2, s_3)) = 0 \qquad (j = 1, 2)$$

Now letting $s_1, s_2, s_3 \to s$ a fixed value and assuming that the circle through $\gamma(s_1), \gamma(s_2), \gamma(s_3)$ has a limiting position $\cancel{\Sigma(s_1, s_2, s_3)}$ $\Sigma(s)$, the point $C(s_1, s_2, s_3)$ must then approach a point $C_\gamma(s)$.

$\Sigma(s)$ is called the osculating circle to γ at $\gamma(s)$ and its center $C_\gamma(s)$ is called the center of curvature.

Then

$$\gamma''(s) \cdot (\gamma(s) - C_\gamma(s)) = -1 \qquad (4)$$

$$\gamma'(s) \cdot (\gamma(s) - C_\gamma(s)) = 0 \qquad (5)$$

Note that if $\gamma''(s) = 0$, $C_\gamma(s)$ doesn't exist.

Let us now assume $\gamma''(s) \neq 0$ and proceed to determine $C_\gamma(s)$.

Since the curve is parametrized by arc length, $\|\gamma'(s)\| = 1$ or $\langle \gamma'(s), \gamma'(s) \rangle = 1$

Differentiating gives the equation $\gamma'(s) \cdot \gamma''(s) = 0$ or $\gamma'(s)$ and $\gamma''(s)$ are orthogonal – thereby providing a basis (~~at $\gamma(s)$~~).

$$\therefore \gamma(s) - C_\gamma(s) = A\gamma'(s) + B\gamma''(s)$$

for certain scalars A and B. Using (4) and (5) ~~we see that $B = -1/\|\gamma(s) - C_\gamma(s)\|^2$~~ and taking dot product with $\gamma'(s)$ gives $A = 0$.

Whereas dot product with γ'' gives

$$\|\gamma''(s)\|^2 B = (\gamma(s) - C_\gamma(s)) \cdot \gamma''(s) = -1 \quad \text{or}$$

$$B = \frac{-1}{\|\gamma''(s)\|^2}$$

Thus

$$C_\gamma(s) = \gamma(s) + \frac{\gamma''(s)}{\|\gamma''(s)\|^2} \qquad (7)$$

and finally $\rho(s) = \|\gamma(s) - C_\gamma(s)\|^2 = \|\gamma''(s)\|^{-2}$

Thus the radius of the osculating circle is $\|\gamma''(s)\|^{-1}$

Def: The curvature $k(s)$ is the reciprocal of the radius of the osculating circle and so

$$k(s) = \|\gamma''(s)\|.$$

(12) Exercise: We have tacitly assumed that for all s_1, s_2, s_3 sufficiently close to s and distinct there passes a circle through $\gamma(s_1)$, $\gamma(s_2)$ and $\gamma(s_3)$. Moreover this circle has a limiting position $\Sigma(s)$ as $s_1, s_2, s_3 \to s$. Discuss this. Clearly such a circle cannot exist if $\gamma''(s) = 0$. Show that if $\gamma''(s) \neq 0$ then the osculating circle exists.

Hint: Try a taylor expansion and neglect higher order terms beyond $\gamma''(s)$.

Well, suppose that $\gamma''(s_0) \neq 0$ and for s_1, s_2, s_3 arbitrarily close to s_0, $\gamma(s_3) - \gamma(s_1)$ and $\gamma(s_3) - \gamma(s_2)$ are collinear (in which case a circle cannot pass through $\gamma(s_1), \gamma(s_2), \gamma(s_3)$) we shall arrive at a contradiction.

WLOG we may assume $s_3 = s_0$ for we can let $s_3 \to s_0$ in the equation

$$(\gamma(s_3) - \gamma(s_1)) \times (\gamma(s_3) - \gamma(s_2)) = 0$$

Second we may assume $s_0 = 0$ and $\gamma(s_0) = 0$ so that by Taylor's theorem

$$\gamma(s) = as + bs^2 + s^3\sigma(s)$$

for all s in a nbd of 0 : $\|a\| = 1$

So $(as_1 + bs_1^2 + s_1^3\sigma(s_1)) \times (as_2 + bs_2^2 + s_2^3\sigma(s_2)) = 0$

$\therefore (a \times b)(s_1 s_2^2 - s_1^2 s_2) + ((a \times \sigma(s_2)) s_1 s_2^3 - (a \times \sigma(s_1)) s_2 s_1^3)$

$+ (b \times \sigma(s_2)) s_1^2 s_2^3 - (b \times \sigma(s_1)) s_1^3 s_2^2 = 0$

Dividing by $s_1 s_2 (s_1 - s_2)$ and letting $s_1, s_2 \to 0$ we get

$$a \times b = 0$$

Note: To deal with the last term, after dividing we get

$$s_1 s_2 \left((b \times \sigma(s_2)) s_2 - (b \times \sigma(s_1)) s_1 \right) / (s_1 - s_2)$$

Take absolute values and apply the MVT.

But $a = \dot\gamma(0)$
$2b = \ddot\gamma(0)$ $\therefore$ $a \perp b$
and $a \times b = 0 \Rightarrow a = 0$ or $b = 0$ which is a contradiction since $\|a\| = 1$ and $\|b\| \neq 0$.

Thus in a sufficiently small neighborhood $N(s_0)$ of ~~$\gamma(s_0)$ where~~ s_0 such that $\gamma''(s_0) \neq 0$ for any three distinct s_1, s_2, s_3, $\gamma(s_1), \gamma(s_2), \gamma(s_3)$ are non collinear.

Next, to show that $\Sigma(s)$ exists (3) implies $\gamma''(v) \cdot (\gamma(v) - C(s_1, s_2, s_3)) = -1$ for a sequence of values $v \to s_0$; and $\gamma'(\tilde v) \cdot (\gamma(\tilde v) - C(s_1, s_2, s_3)) = 0$ for another seq $\tilde v \to s_0$.

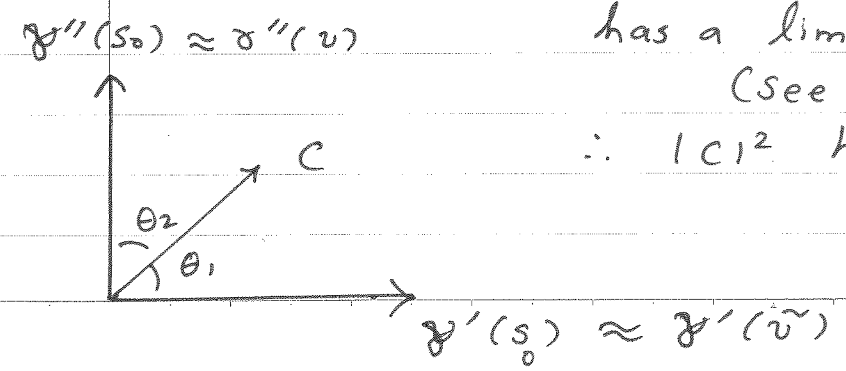
$$\gamma'(\tilde v) \cdot C = \gamma'(\tilde v) \cdot \gamma(\tilde v)$$

$$\gamma''(v) \cdot C = 1 + \gamma''(v) \cdot \gamma(v)$$

$$RHS \to 1 + \gamma'(s_0) \cdot \gamma(s_0) + \gamma''(s_0)\gamma(s_0)$$

and so $|C|^2 (\|\gamma'(\tilde v)\|^2 \cos^2\theta_1 + \|\gamma''(v)\|^2 \cancel{\sin^2\theta} \cos^2\theta_2)$ has a limit as $s_1, s_2, s_3 \to s_0$ (see figure)

$\therefore$ $|C|^2$ has a finite limit as $s_1, s_2, s_3 \to s_0$



Likewise $\rho(s)$ has a finite limit as $s_1, s_2, s_3 \to s_0$.

Formula for Curvature for non unit speed curves:

We now derive the formula for $k(t)$ when the curve is not parametrized by arc length.
Let $s = \phi(t)$ be the arc length function with inverse $t = \psi(s)$

$\sigma = \gamma \circ \psi$ is the arc length reparametrization of γ

$$\sigma'(s) = \frac{\gamma'(\psi(s))}{\phi'(t)} = \frac{\gamma'(\psi(s))}{\|\gamma'(\psi(s))\|}$$

$$\therefore \sigma''(s) = \frac{\gamma''(\psi(s))}{\|\gamma'(t)\|^2} - \frac{\gamma'(t)}{\|\gamma'(t)\|^2} \frac{d}{ds}\|\gamma'(t)\|$$

$$= \frac{\gamma''(t)}{\|\gamma'(t)\|^2} - \frac{\gamma'(t)}{\|\gamma'(t)\|^2} \cdot \frac{\gamma'(t)\cdot\gamma''(t)}{\|\gamma'(t)\|^2} \quad (*)$$

$$= \frac{1}{\|\gamma'(t)\|^4}\left\{\gamma''(t)\|\gamma'(t)\|^2 - \gamma'(t)\,(\gamma'(t)\cdot\gamma''(t))\right\}$$

$$\therefore \|\sigma''(s)\|^2 = \frac{\|\gamma'(t)\times\gamma''(t)\|}{\|\gamma'(t)\|^3}$$

$*$ Well, $\|\gamma'(t)\| = (\gamma'(t)\cdot\gamma'(t))^{1/2}$
Diff. w.r.t s gives $\dfrac{\langle\gamma'(t), \gamma''(t)\rangle}{\|\gamma'(t)\|}\dfrac{dt}{ds}$

$$\therefore\ k(t) = \frac{\|\gamma'(t)\times\gamma''(t)\|}{\|\gamma'(t)\|^3} \quad (8)$$

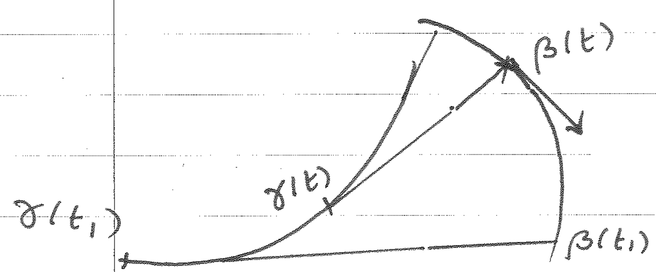
Using equation (7) we get the center of curvature

$$C_\gamma(t) = \gamma(t) + \frac{\gamma''(t)\|\gamma'(t)\|^2 - \gamma'(t)\,(\gamma'(t)\cdot\gamma''(t))}{\|\gamma'(t)\times\gamma''(t)\|^2\,\|\gamma'(t)\|^{-2}} \quad (9)$$

(13) Example: Determine the center of curvature of the parabola $\gamma(t) = (t^2, 2t)$
~~Ans: [illegible]~~

(14) Determine the curve of centeres of curvature for the parabola and the ellipse and discuss their regularity properties.

- Involutes and Evolutes:



Assume that a string of fixed length which is taut along the curve $\gamma(t)$ is allowed to unwind in such a way that the free portion of the string is a line segment tangent to the curve $\gamma(t)$ at the point of contact.
The locus of the free end is called the involute of γ.
To convert this physical description into a mathematical definition we proceed as follows.
Let $\beta(t)$ be the free end of the string and the free portion of the string at time t be the line segment joining $\gamma(t)$ and $\beta(t)$
The tangency condition is that $(\beta(t) - \gamma(t))$ is parallel to $\dot\gamma(t)$

but makes $\measuredangle\ \pi$ with $\dot\gamma(t)$*

$$|\beta(t)-\gamma(t)| - |\beta(t_1)-\gamma(t_1)| = \int_{t_1}^{t} |\gamma'(\lambda)|\,d\lambda = t - t_1$$

or $\frac{d}{dt}|\beta(t)-\gamma(t)| = 1$ or (9)

$$(\beta(t)-\gamma(t))\cdot(\beta'(t)-\gamma'(t)) = |\beta(t)-\gamma(t)| \quad (9')$$

$$= \cancel{t - t_1 + |\beta(t_1)-\gamma(t_1)|}$$

But, ~~if~~ since γ is parametrized by arc length and $\gamma'(t)$ is anti-parallel to $(\beta(t)-\gamma(t))$,

$$(\beta(t)-\gamma(t))\cdot\gamma'(t) = -|\beta(t)-\gamma(t)|$$

$$\therefore (\beta(t)-\gamma(t))\cdot\beta'(t) = 0$$

Again $\beta(t)-\gamma(t) = \lambda\gamma'(t)$ so that

$$\beta'(t)\cdot\gamma'(t) = 0. \quad (10)$$

Def: The involute of $\gamma(t)$ is a curve $\beta(t)$ such that

$\beta(t)-\gamma(t)$ is tangential to $\gamma(t)$

and $\dot\beta(t) \perp \dot\gamma(t)$

With this one can actually work backwards but we prove the result formally.

---

* It would make $\measuredangle$ zero if the reference point $\gamma(t_1)$ is taken on the other end resulting in a change in sign in one of the terms in (9) & ~~(10)~~. (9')

Theorem: $\beta(s)$ is the involute of a unit speed curve α iff for some constant c

$$\beta(s) = \alpha(s) + (c-s)\alpha'(s)$$

proof: If $\beta(s) = \alpha(s) + (c-s)\alpha'(s)$

then $\beta(s)-\alpha(s)$ is tangent to α

and $\dot\beta = (c-s)\ddot\alpha$ so that $\dot\beta\cdot\dot\alpha = 0$

($\because \alpha$ is unit speed)

Conversely let $\beta(s)$ be an involute of α

$\beta(s) = \alpha(s) + \lambda(s)\dot\alpha(s)$ for some scalar function λ

$$\dot\beta = \dot\alpha + \dot\lambda\dot\alpha + \lambda\ddot\alpha$$

Taking dot product with $\dot\alpha$ gives

$0 = 1 + \dot\lambda$ or $\dot\lambda = -1$ and $\lambda(s) = c - s$

the proof is complete.

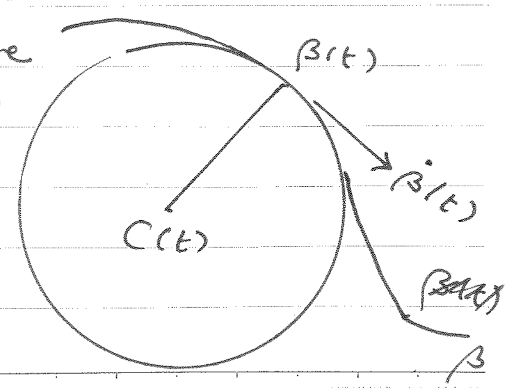
Remark: Note that ~~in~~ the definition of an involute does not make any reference to the parametrization of γ but is expressed solely in terms of geometrical properties that remain invariant under reparametrization.

Def: If $\beta(t)$ is an involute of $\alpha(t)$ then $\alpha(t)$ is called an evolute of $\beta(t)$

Theorem: The evolute of a curve $\beta(t)$ is the ~~locus~~ locus of its centres of curvature

proof: To show $\beta(t)$ is an involute of $C(t)$

~~The condition $\beta(t) - C(t)$ is~~ ~~orthogonal to~~



Well, the osculating circle of β

at the point $\beta(t)$ is certainly tangent to β and so

$$(\beta(t) - c(t)) \cdot \dot\beta(t) = 0 \qquad (11)$$

But by formula (7) (we assume the given curve is parametrized by arc length)

$$\beta(t) - c(t) = - \frac{\ddot\beta(t)}{\|\ddot\beta(t)\|} \quad \text{and} \quad \ddot\beta \cdot \dot\beta = 0$$

so that

$$\dot\beta(t) - \dot c(t) = - \frac{\dddot\beta(t)}{\|\ddot\beta(t)\|} - \ddot\beta(t)\frac{d}{dt}\frac{1}{\|\ddot\beta(t)\|^2}$$

Taking dot product with $\dot\beta(t)$ we get

$$1 - \dot c(t)\cdot\dot\beta(t) = - \frac{\dddot\beta(t)\,\dot\beta(t)}{\|\ddot\beta(t)\|^2}$$

Now differentiate the equation $\ddot\beta(t)\cdot\dot\beta(t) = 0$ and we see $\dot\beta(t)\cdot\dot c(t) = 0$. The second condition has been verified.

But now the condition

$\beta(t) - c(t)$ parallel to $\dot c(t)$ is

equivalent to $\beta(t) - c(t)$ is orthogonal to $\dot\beta(t)$ which follows from (11). The proof is complete.

From exercise (14) we know the evolute of an ellipse.

(15) Determine the involute of a circle.

(16) Determine the involute of a parabola

Exercise: Show that for a non-unit speed curve γ the involute is given by

$$\beta(t) = \gamma(t) + \left(c - \int_{t_0}^{t} |\gamma'(\lambda)|d\lambda\right)\frac{\gamma'(t)}{\|\gamma'(t)\|}$$

Changing t_0 amounts to changing c.

Cycloids (The brachistochrone and tautochrone)

The common cycloid is the curve which is the ~~loce~~ locus of a fixed point on a circle that rolls on a straight line without slipping (call the line "base line")

Let O_t be the instantaneous point of contact of the circle with the x-axis with O_0 = origin. and P_t be the center of the circle at time t and $\gamma(t)$ be the point on the cycloid.

The condition of rolling along a line without slipping may be translated as

length O_0O_t = arc (of the circle) joining $\gamma(\theta)$ and O_t

Let θ be the angle subtended by this arc at the center P_t. and is taken as the parameter along the curve.

Then $s = a\theta$ = length O_0O_t along the x-axis.

The abscissa of $\gamma(\theta)$ is thus

$$a\theta - a\sin\theta$$

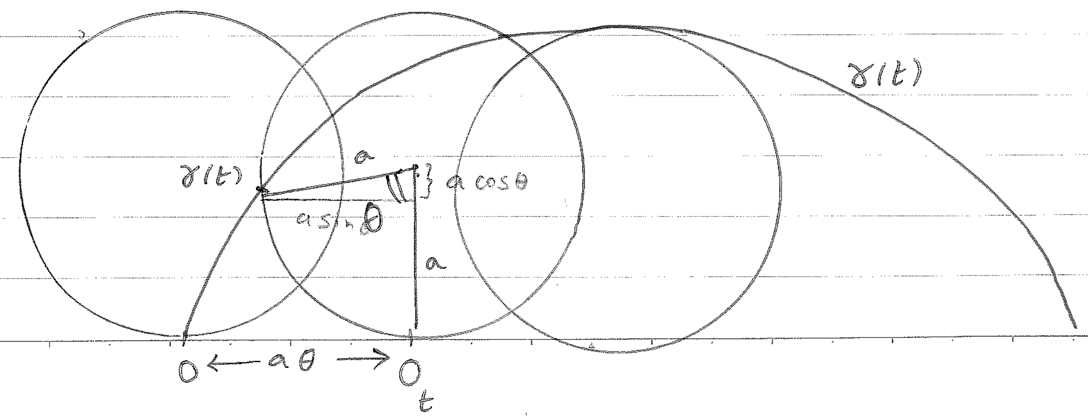
and the ordinate is $a - a\cos\theta$. Thus

$$\gamma : \mathbb{R} \to \mathbb{R}^2 \text{ is given by}$$

$$\gamma(\theta) = (a\theta - a\sin\theta,\ a - a\cos\theta)$$

after a complete revolution the point of the locus is again on the x-axis.

At half time, $\theta = \pi$ and the y-coordinate attains a max $= 2a$.



Show that the area under one arch of the Cycloid is $3\pi a^2 = 3 \times$ Area of gen. circle.

Let us Calculate the arc length of one arch of the Cycloid:

$$= \int_0^{2\pi} \| \gamma'(\theta) \| \, d\theta$$

$$= \int_0^{2\pi} \left((a - a\cos\theta)^2 + a^2 \sin^2\theta \right)^{1/2} d\theta$$

$$= \int_0^{2\pi} a (2 - 2\cos\theta)^{1/2} d\theta$$

$$= 2a \int_0^{2\pi} \sin \theta/2 \, d\theta$$

$$= 8a$$

Note that the parametrization is not the arc length parametrization

Infact $\dfrac{ds}{d\theta} = 2a \sin \theta/2$

So that $\theta = 4 \sin^{-1} \sqrt{\dfrac{s}{8a}}$

Example: Let us Compute the center of Curvature of the cycloid

$$\gamma'(\theta) = (a - a\cos\theta, \; a\sin\theta)$$

$$\gamma''(\theta) = (\cancel{a} \; a\sin\theta, \; a\cos\theta)$$

$\gamma' \times \gamma'' = a^2 \hat{k} (\cos\theta - 1)$ so that

$$\| \gamma'(\theta) \times \gamma''(\theta) \| = a^2 (1 - \cos\theta)$$

$$\gamma'(\theta) \cdot \gamma''(\theta) = a^2 \sin\theta$$

$$\| \gamma'(\theta) \|^2 = 2a^2 (1 - \cos\theta)$$

$$C_\gamma(\theta) = (a\theta + a\sin\theta, \; -a + a\cos\theta)$$

$$k(\theta) = \frac{1}{2\sqrt{2}\, a} \cdot \frac{1}{(1-\cos\theta)^{1/2}} = \frac{1}{4a} \operatorname{cosec} \theta/2 .$$

So that the radius of Curvature $\to 0$ as $\theta \mapsto 0, 2\pi$ (The curve is not regular at $\theta = 0$ and $\theta = 2\pi$) and is ~~minimum~~ maximum at $\theta = \pi$

$$R(\theta) = \text{Rad. Curvature} = 4a \sin \theta/2 .$$

~~Theorem: The Evolute of a cycloid is another Congruent cycloid obtained by reflection along the base line (the line along which the circle rolls without slipping in the def). followed by an improper reparametrization viz reflecting the parameter.~~

~~proof: Reflecting γ along the base line produces $\gamma(\theta) = (a\theta - a\sin\theta, \; -a + a\cos\theta)$~~
~~Now the shift $\theta = 2\pi - \phi$ produces the locus, (2πa − aφ~~

~~unit~~ reparametrization.

~~Proof: Reflection along the x axis produces the locus $(a\theta + a\sin\theta, \; -a + a\cos\theta)$.~~

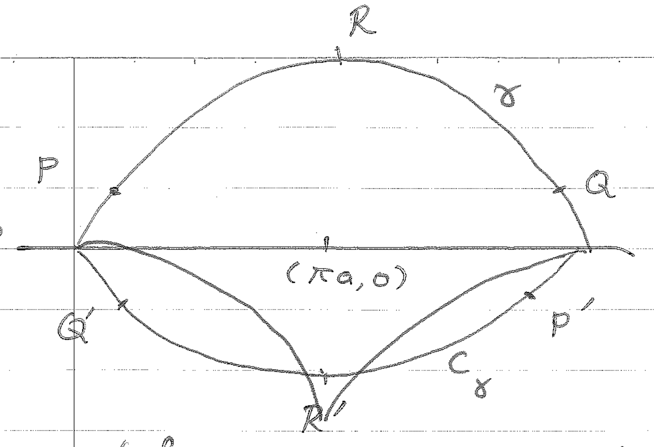
Theorem: The Evolute of a cycloid γ is another Congruent Cycloid obtained from γ by an isometry of $\mathbb{R}^2$ followed by a reparametrization (a linear reversal of the parameter) followed by another isometry of $\mathbb{R}^2$

proof: We shall show that the curves

$$\gamma(\theta) = (a\theta - a\sin\theta, \; a - a\cos\theta)$$

$$C_\gamma(\theta) = (a\theta + a\sin\theta, \; -a + a\cos\theta)$$

$\theta \in \mathbb{R}$ are Congruent.



doesn't seem correct.

~~Consider a point P on γ close to the origin. The center of curvature is a point P' which is a reflection of P about the point $(\pi a, 0)$ directly under the summit R~~

(figure not accurate) on the base line. This suggests the following:

The ~~Base~~ isometry $y \mapsto -y$
$x \mapsto \pi a - x$

transforms $\gamma(\theta) = (a\theta - a\sin\theta,\ a - a\cos\theta)$ into the curve

$$\tilde{\gamma}(\theta) = (a(\pi-\theta) + a\sin(\pi-\theta),\ -a - a\cos(\pi-\theta))$$

The parameter reversal

$$\theta \mapsto \pi - \theta$$ produces the curve

$$\tilde{\tilde{\gamma}}(\theta) = (a\theta + a\sin\theta,\ -a - a\cos\theta)$$

This followed by an ~~reflection plus tran~~ isometry

$$x \mapsto x$$
$$y \mapsto -y \ \cancel{+a}\ -2a$$ produces

$C_\gamma(\theta)$. The proof is complete.

(x, y) ↦ ~~(x,y)~~ $(x+\pi a,\ y-2a)$ transforms $\gamma(\theta)$ to $((\pi+\theta)a - a\sin\theta,\ -a-a\cos\theta)$. Now the parametric shift $\theta \mapsto \theta+\pi$ gives $C_\gamma(\theta)$

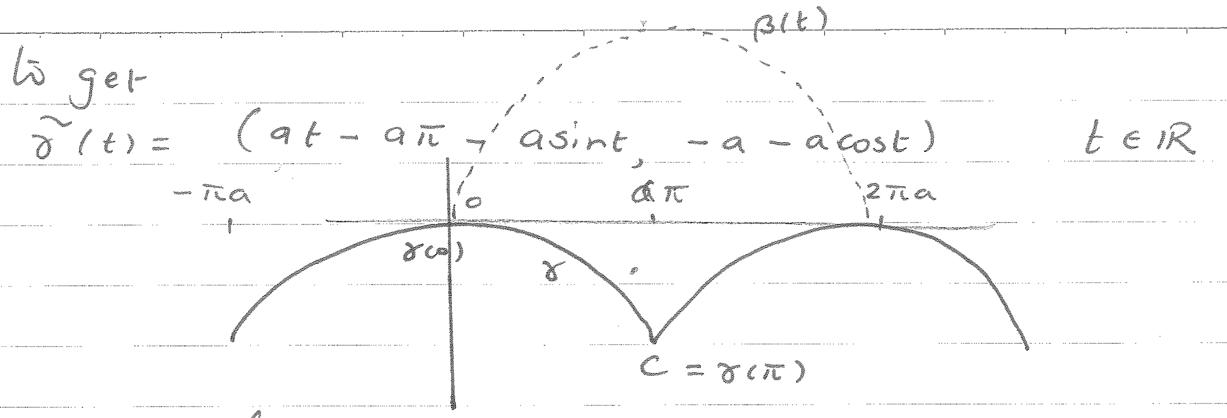
Picture is wrong

Analysis is correct: but can be simplified.

Let us now determine the involute of a cycloid. For convenience we shift the cycloid so that the arches are all tangent to the x-axis and origin is a point of tangency. This is simply achieved by ~~a tr~~ a translation through $(a\pi, +2a)$ i.e subtracting off from $(at - a\sin t,\ a - a\cos t)$ the vector $(a\pi, 2a)$

to get

$$\tilde{\gamma}(t) = (at - a\pi - a\sin t,\ -a - a\cos t) \qquad t \in \mathbb{R}$$



We would like ~~to~~ shift the time parameter so that at time $t=0$ we are at the origin and at time $t=\pi$ at the cusp C.

Thus we put $t \mapsto t - \pi$

$$\gamma(t) = (at + a\sin t,\ -a + a\cos t)$$

$$\beta(t) = \gamma(t) + (c - s(t))\frac{\gamma'(t)}{\|\gamma'(t)\|} \quad : \ 0 \le t \le \pi \text{ say.}$$

$$s(t) = \int_0^t \|\gamma'(\lambda)\|\, d\lambda = 4a\sin\frac{t}{2} \text{ and so}$$

$$\beta(t) = \gamma(t) + \left(c - 4a\sin\frac{t}{2}\right)\left(\cos\frac{t}{2},\ -\sin\frac{t}{2}\right)$$

(note that $\sin t/2 > 0$ on $(0,\pi)$)

Since γ is symmetrical about the line $x = a\pi$ so would the involute (and hence it is enough to restrict t to $(0,\pi)$).

Exercise: Prove the above claim.

put $t=0$ and $\beta(0) = (c, 0)$.

Take $c=0$ so that ~~we start at~~ the involute starts at the origin (which amounts to choosing the length of the string $= 4a$)

Then $\beta(t) = (at - a\sin t,\ a - a\cos t)$

We see that the involute is again a congruent cycloid for this specific choice of c.

If we take a different value of c our curve will be

$$\tilde{\beta}(t) = (at - a\,\overset{\sin t}{\cancel{\cos t}},\ a - a\cos t) + \alpha \frac{\gamma'}{\|\gamma'\|}$$

Investigate this curve. Note that the involute $\beta(t)$ is dragged down by a ~~unit~~ vector of constant length α.
The direction of drag is along the normal to β by the property of involute (!)
Would $\tilde{\beta}(t)$ be a cycloid ?

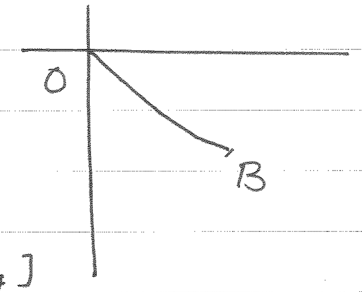
Now $\beta(\pi) = (a\pi, 2a)$ and so the point $\beta(\pi)$ lies vertically above the cusp C and at a distance $4a$. = length of the string = length of the half arch along the cycloid from origin to C.
The above example will play a crucial rôle in our discussion of the tautochrone property of the cycloid.

The brachistochrone property of a cycloid.

Brachistos = Shortest; Chronos = time

Let us consider a curve joining two points $A(x, y_1)$ and $B(x_2, y_2)$ such that $y_2 < y_1$. A particle slides down along the curve such that the time of descent is least among all choices of curves joining A and B. To determine the shape of the curve.

This is a problem in calculus of variations.
Assume A is the origin and B is in the fourth quadrant



Let us take the curve

$$\gamma(x) = (x, f(x)) \text{ for}$$

some function f of x on $[0, x_1]$

The instantaneous kinetic energy is then $\frac{1}{2}\left(\frac{ds}{dt}\right)^2$ (mass of the particle = 1) ~~and the total kinetic energy~~

$$= \frac{1}{2}(1 + f'(x)^2)\left(\frac{dx}{dt}\right)^2$$

which must be equal to the loss in potential energy $-g\,\overset{}{\cancel{y}}\,f(x)$ (recall that $y < 0$)

$$\therefore \frac{(1 + f'(x)^2)^{1/2}}{(-y)^{1/2}}\frac{dx}{dt} = \sqrt{2g} \quad \text{or}$$

$$\sqrt{2g}\, T_{\text{brachistochrone}} = \int_0^{x_1} \frac{\sqrt{1 + (f'(x))^2}\, dx}{\sqrt{-\cancel{y}f(x)}}$$

The problem is then to select the curve γ such that

$$I(\gamma) = \int_0^{x_1} \frac{(1 + f'(x)^2)^{1/2}\, dx}{\sqrt{-\cancel{y}f(x)}} \text{ is least}$$

(The conv. of the integral is taken up at the end)

Let $\mathcal{F}$ be the class of all smooth curves joining A to B and we have a map

Called a "functional" $I : \mathcal{F} \to \mathbb{R}$ which we seek to minimize.

Let $\mu = \inf_{\gamma \in \mathcal{F}} I(\gamma)$

and $\exists$ a seq $\{\gamma_n\}$ of curves in $\mathcal{F}$ s.t.

$I(\gamma_n) \to \mu$

but in general (γ_n) may not converge in $\mathcal{F}$ in any reasonable sense. However in the case at hand one can show the existence of a minimizer $\gamma = \lim_n \gamma_n$ by using the so called Ascoli Arzela theorem. However such an analysis is beyond the scope of this course.

We proceed somewhat heuristically.

In general the linear functional I assumes the form

$$I(\gamma) = \int_\alpha^\beta F(x, \gamma, \gamma')\,dx$$

where $\gamma : [\alpha, \beta] \to \mathbb{R}^n$ is an admissible curve and $F : [\alpha, \beta] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is smooth.

If γ_0 is a ~~minizer~~ minizer then we take a curve

$$\gamma_0 + \epsilon\sigma : [\alpha, \beta] \to \mathbb{R}^n$$

satisfying $\sigma(\alpha) = \sigma(\beta) = 0$ (so that the curves $\gamma_0 + \epsilon\sigma$ all have the same end pts)

Performing a Taylor Series expansion for the function

$$F(x, y, z) = F(x_0, y_0, z_0) + \partial_1 F(x, y_0, z_0)(y - y_0) + \partial_2 F(x, y_0, z_0)(z - z_0) + \frac{1}{2}\left(\partial_{11}^2 F(x, y_0, z_0)(y - y_0)^2 + 2\partial_{12} F(x, y_0, z_0)(y - y_0)(z - z_0) + \partial_{22}^2 F(x, y_0, z_0)(z - z_0)^2\right) + R$$

where R is the remainder and

$$|R| \leq A\left(|y - y_0|^2 + |z - z_0|^2\right)^{3/2}.$$

$\partial_1 F$ refers to $\frac{\partial F}{\partial y}$, and $\partial_2 F = \frac{\partial F}{\partial z}$

Write $I(\gamma_0 + \epsilon\sigma) = \phi(\epsilon)$ so that for a minimizer γ_0, $\phi(0) = 0$, $\phi(\epsilon) \geq \phi(0)$ for all ϵ close to the origin.

$$\therefore \phi(\epsilon) = \phi(0) + \epsilon \int_\alpha^\beta \partial_1 F(x, \gamma_0, \gamma_0')\,\sigma(x)\,dx + \epsilon \int_\alpha^\beta \partial_2 F(x, \gamma_0, \gamma_0')\,\sigma'(x)\,dx + \epsilon^2 Q(x) + \epsilon^3 R(x)$$

$Q(x)$ = quadratic terms.

We perform an integration by parts in the $\partial_2 F$ term

$$\int_\alpha^\beta \partial_2 F(x, \gamma_0, \gamma_0')\,\sigma'(x)\,dx = -\int_\alpha^\beta \frac{d}{dx}\left(\partial_2 F(x, \gamma_0, \gamma_0')\right)\sigma(x)\,dx$$

by virtue of the boundary conditions $\sigma(\alpha) = \sigma(\beta) = 0$ and we have

$$(*) \quad \phi(\epsilon) = \phi(0) + \epsilon \int_\alpha^\beta \left[\partial_1 F(x, \gamma_0, \gamma_0') - \frac{d}{dx}\left(\partial_2 F(x, \gamma_0, \gamma_0')\right)\right]\sigma(x)\,dx + \epsilon^2 Q(x) + \epsilon^3 R(x)$$

The Nec. Cond $\phi'(0) = 0$ now gives

$$\int_\alpha^\beta \left[\partial_1 F(x, \gamma_0, \gamma_0') - \frac{d}{dx}\partial_2 F(x, \gamma_0, \gamma_0')\right]\sigma(x)\,dx = 0$$

Ex: Suppose f is a continuous function such that for all smooth functions (once diff) g ~~the~~ with

Cont

$g(\alpha)=g(\beta)=0, \quad \int_\alpha^\beta f(t)g(t)\,dt=0$

then $\quad f(t)\equiv 0 \quad$ on $[\alpha,\beta]$.

Thus, we get the Necessary Condition for the minimizer γ_0 namely

$$\partial_1 F(x,\gamma_0,\gamma_0') - \frac{d}{dx}\,\partial_2 F(x,\gamma_0,\gamma_0') = 0$$

or

$$\frac{\partial F}{\partial y}(x,\gamma_0,\gamma_0') - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x,\gamma_0,\gamma_0')\right) = 0$$

which is called the Euler Lagrange Equation for the variational problem.

For the Case of the brachistochrone

$$F(x,y,y') = (1+y'^2)^{1/2}/\sqrt{-y}$$

and a simple calculation gives the Euler Lagrange eqn.

$$\frac{1}{2}\sqrt{1+y'^2}\,(-y)^{-3/2} - \frac{d}{dx}\left(\frac{y'}{\sqrt{-y}\,\sqrt{1+y'^2}}\right) = 0$$

$$\therefore \frac{1}{2}\sqrt{1+y'^2}\,(-y)^{-3/2} - \frac{1}{2}\frac{(-y)^{-3/2}\,y'^2}{(1+y'^2)^{1/2}}$$

$$-\frac{y''}{\sqrt{-y}\,\sqrt{1+y'^2}} + \frac{y'}{\sqrt{-y}}\,\frac{y'y''}{(1+y'^2)^{3/2}} = 0$$

$$\therefore \frac{1}{2}\frac{(-y)^{-3/2}}{\sqrt{1+y'^2}} + \frac{y''}{\sqrt{-y}\,(1+y'^2)^{3/2}} = 0$$

$$2y\,y'' + (1+y'^2) = 0$$ which is an ODE for the desired curve. The initial Condition is $y(0)=0$

~~$y(0)\neq 0$ singular~~

~~The second condition is given by the fact that the particle starts from rest at the origin. Note that the ODE is singular at $x=0$ and finiteness~~

Note that the ODE is singular at the origin. There is no hope of finding a solution of class C^2 for by putting $x=0$ in the equation we get
$$1+y'^2=0.$$

put $\quad y'=u<0$ (since y decreases as x increases)

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\frac{dy}{dx} = \frac{du}{dy}\,u$$

$$\therefore \quad 2yu\frac{du}{dy} + (1+u^2) = 0$$

$$\therefore \quad y(1+u^2) = c \qquad (c<0)$$

$$\therefore \quad \frac{dy}{dx} = u = -\sqrt{\frac{c-y}{y}} \qquad \text{put } y = c\sin^2\theta$$

$$\therefore \quad 2c\sin\theta\,\cancel{\cos\theta}\cdot\frac{\sin\theta}{\cancel{\cos\theta}}\,d\theta = -dx$$

$$\therefore \quad -x+\tilde{a} = c\theta - \frac{c}{2}\sin 2\theta$$

$$\therefore \quad 2x - 2\tilde{a} = (-c)(2\theta - \sin 2\theta) \quad \text{and}$$

$$y = c\sin^2\theta = \frac{c}{2}(1-\cos 2\theta)$$

$\tilde{a}=0$ since the curve passes through the origin and we get

$$2x = (-c)(2\theta - \sin 2\theta)$$
$$2y = c\,(1-\cos 2\theta)$$

Which is a cycloid in the 4 th Quadrant.

Theorem: The Cycloid is the brachistochrone.

Two points remain to be discussed

(i) the meaning of the integral $\displaystyle\int_0^{x_1} \frac{\sqrt{1+y'^2}\,dx}{\sqrt{-y}}$ near the origin

(ii) Is the solution obtained a minimum or merely a stationary solution (we have only applied the nec. conditions for local minima)

Let us take up the second point and go back to the integral (*)

$$\phi(\epsilon) = \phi(0) + \epsilon^2 Q(x, \gamma_0, \gamma_0') + R$$

(Since the first order terms have vanished)

A direct calculation gives*

$$Q = \frac{1}{2}\int_\alpha^\beta \left(\frac{\partial^2 F}{\partial y^2}\sigma^2 + 2\frac{\partial^2 F}{\partial y \partial y'}\sigma\sigma' + \frac{\partial^2 F}{\partial y'^2}\sigma'^2 \right) dt$$

$$= \frac{1}{2}\int_\alpha^\beta \Big[\left(\frac{3}{4}\operatorname{cosec}^6\frac{\theta}{2}\right)(\sigma(\theta))^2 - \cos\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}\,\sigma(\theta)\sigma'(\theta) + (\sin^4\theta/2)(\sigma'(\theta))^2 \Big] d\theta$$

The quadratic form under the integral sign is positive definite so that $Q > 0$.

Now $\phi(\epsilon) > \phi(0)$ for all ϵ small enough which confirms that γ_0 is indeed a minima.

$$* \quad \frac{\partial^2 F}{\partial z^2} = \frac{1}{\sqrt{-y}}\frac{1}{(z^2+1)^{3/2}} = \sin^2\theta/2 \quad ; \quad z = \frac{dy}{dx} = \frac{y_\theta}{x_\theta}$$

$$\frac{\partial^2 F}{\partial y^2} = \cancel{\tfrac{3}{4}\operatorname{cosec}}\ \frac{3}{4}\frac{(1+z^2)^{1/2}}{(-y)^{5/2}} = 3/4 \operatorname{cosec}^6\theta/2$$

$$\partial^2_{yz} F = \frac{1}{2}\frac{z}{\sqrt{1+z^2}}(-y)^{-3/2} = -1/2\cos\theta/2\operatorname{cosec}^2\theta/2.$$

Since the curve starts at the origin

$x(t) = at + bt^2 + \dots$

$y(t) = \alpha t + \beta t^2 + \dots$ as functions of time.

Also since the particle starts from rest $\dot{x}(0) = \dot{y}(0) = 0$ so that $a = \alpha = 0$

The acceleration in the vertical direction $\neq 0$

$\therefore \beta \neq 0$

$\therefore$ the factor $\dfrac{dx}{\sqrt{y}}$ in the integral equals

$$\frac{(2bt + 3ct^2 + \dots)dt}{t\sqrt{\beta}(1+\lambda t + \dots)^{1/2}} = \frac{(2b + 3ct + \dots)dt}{\sqrt{\beta}(1+\lambda t+\dots)^{1/2}}$$

which makes perfect sense.

Exercise: In the above discussion we have assumed that F is a function from $\mathbb{R}^3 \to \mathbb{R}$ with γ depending on x and the integrand in the objective function I being $F(x, \gamma(x), \gamma'(x))$

Discuss what happens if ~~F: $\mathbb{R}^3 \to \mathbb{R}$~~ $F: I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with the class $\mathcal{F}$ consisting of curves $\gamma: I \to \mathbb{R}^n$ with $\gamma(\alpha) = a$, $\gamma(\beta) = b$ fixed and $\gamma = (\gamma_1(t), \dots, \gamma_n(t))$. What form will the perturbed curve $\gamma_0 + \epsilon\sigma$ assume? ~~how many equa~~ What will be the nature of the Euler Lagrange equations?

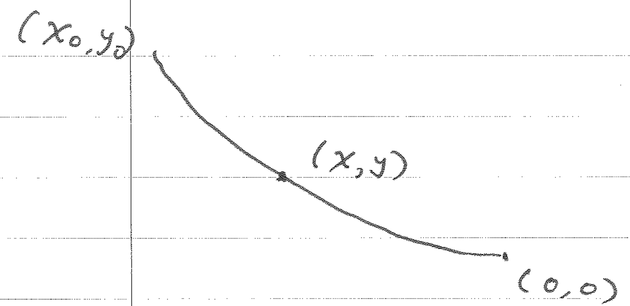
Discuss the isoperimetric problem: Of all closed curves γ with a given perimeter the circle encloses the ~~least area~~ greatest area

Imitate the method of Lagrange multipliers.

The tautochrone property of the cycloid.

A wire frame has to be constructed such that a bead sliding down the wire reaches the bottom most point in a time T that is independent of the height of release.

Let the bottommost point be the origin and let (x_0, y_0) be the point of release and (x, y) be any intermediate point.



The Kinetic energy at (x,y) is $\frac{1}{2} m \left(\frac{ds}{dt}\right)^2$

Which must balance the change in potential energy $mg(y_0 - y)$ so that

$$\left(\frac{ds}{dt}\right)^2 = 2g(y_0 - y)$$

$$\frac{ds}{dt} = -\sqrt{2g}\,\sqrt{y_0 - y} \quad *$$

Regard s as a function of y say $F(y)$ and

$$\frac{F'(y)\,dy}{\sqrt{y_0 - y}} = -\sqrt{2g}\,dt$$

$$\therefore \int_0^{y_0} \frac{F'(y)\,dy}{\sqrt{y_0 - y}} = -\sqrt{2g}\int_T^0 dt$$

$$\text{Or} \quad T = \frac{1}{\sqrt{2g}} \int_0^{y_0} \frac{F'(y)\,dy}{\sqrt{y_0 - y}} \qquad (**)$$

* s is the arc length measured from the origin.

Thus we have to solve an integral ~~equation~~ equation for $F(y)$.

Equation $(**)$ is called an integro-differential equation of convolution type.

We use Laplace transforms to solve this.

Taking the Laplace transform of $(**)$

$$\frac{T}{\lambda} = \frac{1}{\sqrt{2g}}\,\mathcal{L}(F')\,\frac{\sqrt{\pi}}{\sqrt{\lambda}}$$

$$\therefore \frac{T}{\sqrt{\lambda}} = \sqrt{\frac{\pi}{2g}}\,\mathcal{L}(F')$$

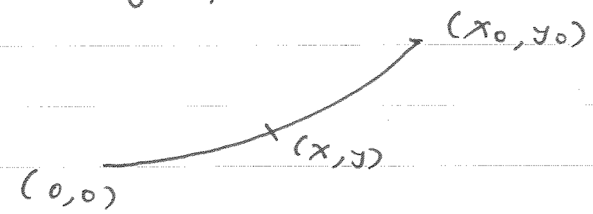
$$\therefore \mathcal{L}(F') = \sqrt{\frac{2g}{\pi\lambda}}\,T$$

$$\therefore F' = \sqrt{\frac{2g}{\pi}}\,T\,\frac{1}{\sqrt{\pi}\sqrt{y}} = \frac{\sqrt{2g}}{\pi}\frac{T}{\sqrt{y}}$$

$$\therefore 1 + \left(\frac{dx}{dy}\right)^2 = \frac{\sqrt{2g}}{\pi}\frac{T}{y}$$

$$\therefore dx = \pm\sqrt{\frac{-y+a}{y}}\,dy \;; \quad \left(a = \frac{\sqrt{2g}}{\pi}\,T\right)$$

(Since y decreases as x increases in the picture but the situation would be reversed in the following picture



We take both signs)

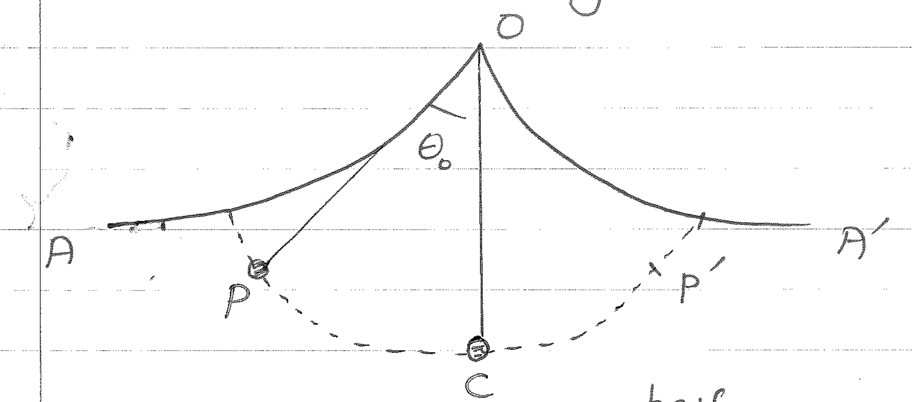
Integrating as before

$$2y = a(1 - \cos 2\theta)$$
$$2x = \pm a(2\theta - \sin 2\theta)$$

Which is a cycloid.

Huygen's Isochronus Pendulum:

Recall that the period of a simple pendulum depends upon the amplitude. C. Huygens constructed in an isochronus pendulum whose period is independent of the amplitude of the swing. For an interesting history of the subject see the book of R. Dugas



Consider two successive half arches AO and OA' of a cycloid and assume that a pendulum is suspended from O and OC is the mean position. If the pendulum is released from say p then the bob traces an involute of a cycloid PCP' and swings with amplitude θ_0. But then due to the tautochrone property of the cycloid the period would be independent of the amplitude θ_0 thereby providing an isochronous pendulum

Exercise. For a simple pendulum determine the period as a function of the amplitude.

Space Curves and the Frenet Serret Frame

A space curve is a smooth map $\gamma : I \to \mathbb{R}^3$ where I is an open interval in $\mathbb{R}$.
As usual the arc length function is given by
$$s(t) = \int_{t_0}^{t} |\gamma'(\lambda)| d\lambda$$
and one can reparametrize γ by its arc length.

For a curve parametrized by its arc length, $\vec{t} = \gamma'(s)$ is the unit tangent vector at the point $\gamma(s)$
Now $\langle \vec{t}(s), \vec{t}(s) \rangle = 1$
$$\Rightarrow \frac{d\vec{t}}{ds} \perp \vec{t}(s)$$
~~We also~~ Thus $\frac{d\vec{t}}{ds}$ is a normal vector to the space curve but unlike the case of a plane curve there are infinitely many normals to a space curve.
The unit vector $\vec{n}(s) = \frac{t'}{\| t' \|}$
i.e. $\vec{n}(s) = \frac{\gamma''(s)}{\| \gamma''(s) \|}$
is called the principal normal to the curve. It exists only at those points $\gamma(s)$ for which $\gamma''(s) \neq 0$.
It may be noted that when γ is a plane curve $\gamma''(s)$ is parallel to the

radius vector $\gamma(s) - C_\gamma(s)$ joining $\gamma(s)$ and the center of curvature.

Now let us assume $\gamma''(s) \neq 0$ and s_1, s_2 be two values of the parameter close to s

$(\gamma(s_1) - \gamma(s)) \times (\gamma(s_2) - \gamma(s))$ is then a vector $\perp$ to the plane $\Pi(s, s_1, s_2)$ through the three points $\gamma(s)$, $\gamma(s_1)$, $\gamma(s_2)$. The circle through $\gamma(s)$, $\gamma(s_1)$, $\gamma(s_2)$ also lies on this plane
$$\gamma(s_1) - \gamma(s) = (s_1 - s)\dot{\gamma}(s) + \frac{(s_1-s)^2}{2!}\ddot{\gamma}(s) + \cdots$$
$$\gamma(s_2) - \gamma(s) = (s_2 - s)\dot{\gamma}(s) + \frac{(s_2-s)^2}{2!}\gamma''(s) + \cdots$$
$\therefore (\gamma(s_1) - \gamma(s)) \times (\gamma(s_2) - \gamma(s))$
$$= (s_1 - s)(s_2 - s)\left\{ \frac{(s_1 - s_2)}{2} \dot{\gamma}(s) \times \ddot{\gamma}(s) + \cdots \right\}$$
Dividing by $(s_1 - s)(s_2 - s)(s_1 - s_2)$ and letting $s_1, s_2 \to s$ we see that ~~that~~ there is a limiting value to the unit-normal vector to $\Pi(s, s_1, s_2)$ namely
$$\frac{\dot{\gamma}(s) \times \ddot{\gamma}(s)}{\| \dot{\gamma}(s) \times \ddot{\gamma}(s) \|}$$
The lines through $\gamma(s_1)$ and $\gamma(s)$, $\gamma(s_2)$ and $\gamma(s)$ also lie on $\Pi(s, s_1, s_2)$ which means that the plane $\Pi(s, s_1, s_2)$ has a limiting position $\Pi(s)$ as the plane containing the line
$$\{ \gamma(s) + \lambda \vec{t}(s) \,/\, \lambda \in \mathbb{R} \}.$$
and orthogonal to $\frac{\dot{\gamma}(s) \times \ddot{\gamma}(s)}{\| \dot{\gamma}(s) \times \ddot{\gamma}(s) \|}$

Think we used the fact $\dot{\gamma}, \ddot{\gamma}$ is a basis — what about here? Examine this pt with care

The plane $\Pi(s)$ is called the osculating plane of γ at the point $\gamma(s)$.

Now if we refer to the Chapter on plane curve and $C(s, s_1, s_2)$ is the circle through $\gamma(s), \gamma(s_1), \gamma(s_2)$ then

(i) $C(s, s_1, s_2)$ lies on $\Pi(s, s_1, s_2)$

(ii) The conditions

$$\gamma''(s) \cdot (\gamma(s) - C_\gamma(s)) = -1$$
$$\gamma'(s)\,(\gamma(s) - C_\gamma(s)) = 0$$

would still hold for the limiting position of $C(s, s_1, s_2)$ with $C_\gamma(s)$ being ~~its~~ the center of the limiting circle.

Again, $\gamma'(s), \gamma''(s), \gamma'(s) \times \gamma''(s)$ forms a basis and so

$$\gamma(s) - C_\gamma(s) = A\,\gamma'(s) + B\gamma''(s) + C\,(\gamma'(s) \times \gamma''(s))$$

($C = 0$ obviously.)

Dot prod. with $\gamma'(s)$ gives $A = 0$

Dot prod with $\gamma''(s)$ gives $B = -\dfrac{1}{\|\gamma''(s)\|^2}$

But clearly $C(s, s_1, s_2) \subset \Pi(s, s_1, s_2)$ so that $\gamma(s) - C_\gamma(s)$ would be orthogonal to the normal $\gamma'(s) \times \gamma''(s)$ to $\Pi(s)$

So $C = 0$ and we get again

$$C_\gamma(s) = \gamma(s) + \frac{\gamma''(s)}{\|\gamma''(s)\|^2}$$

for the center of the osculating circle (which now lies on the osculating plane)

The principal normal is along the radius vector joining $\gamma(s)$ and $C(s)$.

The radius of the osculating circle is $\dfrac{1}{\|\gamma''(s)\|}$ and its reciprocal is called the curvature $k(s)$ of the curve.

Thus

$$\vec{t}(s) = \gamma'(s)$$
$$k(s) = \|\gamma''(s)\|$$
$$\vec{n}(s) = \frac{\gamma''(s)}{\|\gamma''(s)\|}$$

(normal pointing towards the centre of curvature) and (lying on or) parallel to the osculating plane.

The normal vector $\gamma'(s) \times \dfrac{\gamma''(s)}{\|\gamma''(s)\|}$

$$\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$$ is called the binormal to the curve.

Note that $\vec{b}(s)$ is both normal to the curve and the osculating plane.

$\vec{n}(s)$ is called the principal normal to the curve.

The plane spanned by $\vec{t}, \vec{n}$ is the osculating plane

The plane spanned by $\vec{n}, \vec{b}$ is called the normal plane

" " " $\vec{b}, \vec{t}$ is called the rectifying plane

~~Def~~: $\dfrac{d}{ds}\vec{b}(s) \perp \vec{b}(s)$

Thus $\dfrac{d}{ds}\vec{b}(s) = \alpha\,\vec{t}(s) + \beta\,\vec{n}(s)$

for certain scalars α, β.

But then $\vec{b} \cdot \vec{t} = 0$

$\Rightarrow b' \cdot t + b \cdot t' = 0$

$\Rightarrow b' \cdot t + b \cdot n = 0$

$\Rightarrow b' \cdot t = 0$

Thus $\alpha = 0$ and $b' = -\tau\,\hat{n}(s)$

The scalar function τ is called

$$\tau = \hat{b} \cdot \frac{d\hat{n}}{ds}$$

the torsion to the curve.
It measures the rate of change of the direction of $\hat{b}$ i.e. the rate at which $\gamma(s)$ deviates from being planar.

Thm: $\gamma(s)$ is a plane curve iff $\vec{b}$ is a constant vector. parametrized by arc length

proof: If γ is a plane curve then γ'' lies in the plane of γ and so does γ'.
Thus $\dfrac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|}$ is constant viz the unit vector to that plane.
Conversely if $\hat{b}$ is constant then look at

$$f(s) = (\gamma(s) - \gamma(0)) \cdot \hat{b}$$
$$f'(s) = \gamma'(s) \cdot \hat{b} = t \cdot b = 0$$

or $f(s)$ is constant. But $f(0) = 0$

$$\therefore (\gamma(s) - \gamma(0)) \cdot \hat{b} = 0 \quad \forall s$$

i.e. $\gamma(s)$ lies on the plane passing through $\gamma(0)$ and $\perp$ to $\hat{b}$

Theorem:
$$n'(s) = -k(s)\, t + \tau b$$
$$b'(s) = -\tau(s)\, n$$
$$t'(s) = k(s)\, n$$

proof: Only the first equation needs proof.
$$n' \cdot n = 0 \quad (\because n \cdot n = 1)$$
$$\therefore n' = \lambda t + \mu b \text{ for some scalars } \lambda, \mu$$

$$\therefore \lambda = n' \cdot t = n' \cdot t + n \cdot t' - n \cdot t'$$
$$= (n \cdot t)' - n \cdot kn = -k$$

$$\mu = n' \cdot b$$
$$= (n \cdot b)' - n \cdot b' = \tau.$$

Schematically

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

The formulas stated in the theorem are called the Serret-Frenet formulas.

Ex. Compute the curvature & torsion for the circular helix

Fundamental theorem for curves in $\mathbb{R}^3$:

Let k and τ be two functions on $(a,b) \to \mathbb{R}$ which are continuous and $k > 0$.
Then $\exists!$ unit speed curve $\gamma : (a,b) \to \mathbb{R}^3$ whose curvature and torsion functions are k and τ respectively.
The uniqueness is in the sense that any two such curves differ by a rigid Euclidean motion.

proof: Consider the system of ordinary diff. equations

$$v_1' = k v_2$$
$$v_2' = -k v_1 + \tau v_3$$
$$v_3' = -\tau v_2$$

where v_1, v_2, v_3 are vector valued functions
This is a linear system of 9 ODEs which may be written as a matrix system

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \Lambda(s) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

writing (contrary to usual conventions) the vectors

v_1, v_2, v_3 as rows.
The system of ODEs has a unique solution for any given initial conditions which we take to be an orthonormal frame:

i.e $\begin{bmatrix} v_1(s_0) \\ v_2(s_0) \\ v_3(s_0) \end{bmatrix}$ is an orthogonal matrix

with unit determinant.
Denote the skew-symm matrix

$$\begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \text{ by } \Lambda(s)$$

Then consider the solution $M(s)$ of the system of ODEs $\dot{M} = \Lambda M$
with $M(s_0)$ an orthogonal matrix with $\det = 1$.

Claim: $MM^T = I \quad \forall s \in (a,b)$
Well,

$$\frac{d}{ds}(MM^T) = \dot{M}M^T + M\dot{M}^T$$
$$= \Lambda MM^T + M(\Lambda M)^T$$
$$= \Lambda MM^T + MM^T\Lambda^T$$
$$= \Lambda MM^T - MM^T\Lambda$$

So MM^T satisfies a linear ODE
$$\dot{X} = \Lambda X - X\Lambda$$
with initial cond. I
But the constant function I also satisfies the same IVP
$$\therefore \quad MM^T = I \quad \forall s.$$

* and same initial values $\gamma_1(s_0) = \gamma_2(s_0)$

Define $\gamma(s) = \int_{s_0}^{s} v_1(\lambda)\,d\lambda$

Then $\|\dot{\gamma}(s)\| = \|v_1\| = 1$ so $\gamma : (a,b) \to \mathbb{R}^3$ is a unit speed curve.

$$\dot{\gamma} = v_1 \qquad \text{(unit tangent vector)}$$
$$\ddot{\gamma} = \dot{v}_1 = k v_2$$
$$\therefore \quad \|\ddot{\gamma}\| = k$$
so that k is indeed the curvature of γ.

The principal normal is v_2
Next $\cancel{\dot{v}_2 = -k v_1 + \tau v_3}$, M orth with $\det = 1$
$\Rightarrow v_3 = v_1 \times v_2$ which shows that v_3 is the binormal and $\dot{v}_3 = -\tau v_2$
$\Rightarrow \tau$ is the torsion of γ

Suppose γ_1, γ_2 are two curves with the same
* k and τ then ~~both satisfy the same ODE~~
$\cancel{\dot{\gamma}_1 = \Lambda\gamma_1}$
$\cancel{\dot{\gamma}_2 = \Lambda\gamma_2}$
$\therefore \cancel{(\gamma_1 - \gamma_2)^{\cdot} = \Lambda(\gamma_1 - \gamma_2)}$
~~if M_1, M_2 are the moving frames for them then~~
$\cancel{(M_1 - M_2)^{\cdot} = \Lambda(M_1 - M_2)}$
~~by Serret-Frenet equations.~~
By Serret-Frenet we have the pair of systems
$$\dot{M}_1 = \Lambda M_1 \ ; \quad \dot{M}_2 = \Lambda M_2$$
But $M_1(s_0) = M_2(s_0)P$ for some orthogonal P with $\det P = 1$
$$\therefore (M_1 - M_2 P)^{\cdot}$$
$$= \Lambda M_1 - \Lambda M_2 P = \Lambda(M_1 - M_2 P)$$
which shows that $M_1 \equiv M_2 P$

Def (Helix) A curve $\gamma : (a,b) \to \mathbb{R}$ of unit speed is called a general helix if $\exists$ a constant vector v such that ~~...~~

$$\langle \dot\gamma, v \rangle \text{ is const}$$

(The cond is $\langle \frac{\dot\gamma}{\|\dot\gamma\|}, v \rangle$ const when γ is not a unit speed curve)

$\vec{v}$ is called the axis of the helix.

Ex: $\gamma(s) = (\cos s, \sin s, s)$

$\dot\gamma(s) = (-\sin s, \cos s, 1)$

$\vec{v} = (0, 0, 1)$

For this helix, $k = 1/\sqrt{2}$,

$n = (-\cos s, -\sin s, 0)$

so that

$\frac{dn}{ds} = (\sin s, -\cos s, 0)$ and

$\tau = \frac{dn}{ds} \cdot b = \frac{dn}{ds} \cdot (t \times n)$

$= 1/\sqrt{2}$ (Check)

Show that if γ is a curve in $\mathbb{R}^3$ for which the curvature and torsion are both constant then γ is a cylinderical* helix

Hint: Use the fund. thm.

* i.e a helix drawn on a right circular cylinder.

Thm: A unit speed curve $\gamma : (a,b) \to \mathbb{R}^3$ (with $k > 0$) is a general helix iff for some constant c

$$\tau(s) = c\, k(s)$$

pf. Let γ be a helix with axis v so that

$t \cdot v = $ const.

$\therefore\ t' \cdot v = 0$ or $k\hat{n} \cdot v = 0$

$\therefore\ \hat{n} \perp v$

We may certainly assume $\|v\| = 1$

$\therefore\ v = \lambda \hat{t} + \mu \hat{b}$

$0 = \dot\lambda t + \dot\mu b + \lambda k n - \mu \tau n$

$\therefore\ \lambda, \mu$ are constants and

$\lambda k - \mu \tau = 0$

Now $\lambda^2 + \mu^2 = 1$ $\therefore\ \mu \neq 0$ and $\tau = c k$.

Conversely if

$\tau(s) = c\, k(s)$ for some const. c

Then $\tau \hat{n} = c k \hat{n}$

$\therefore\ -\frac{d}{ds}\hat{b} = c \frac{d}{ds}\hat{t}$

$\therefore\ \hat{b} + c\hat{t} = $ const $= v$ say

Then $v \cdot \hat{t} = c$ const showing that the curve is a general helix.

Formula for curvature and torsion for curves not parametrized by arc length.

For a curve $\gamma : (a,b) \to \mathbb{R}^3$ not necessarily of unit speed, the curvature and torsion are given by

$$k(t) = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}; \quad \tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2}$$

Let $\gamma(t) = \gamma(\sigma(s))$ where σ is the inverse of the arc length function.

$$\dot\gamma(t) = (\gamma\circ\sigma)^{\cdot}(s)\ \dot\sigma(s) = t(s)\,\dot\sigma(s)$$

$$\therefore\ \ddot\gamma(t) = (\gamma\circ\sigma)^{\cdot\cdot}(s)\,(\dot\sigma(s))^2 + (\gamma\circ\sigma)^{\cdot}(s)\,\ddot\sigma(s)$$

$$= k(s)\,n\,\dot\sigma(s)^2 + \cdots$$

$$\therefore\ \ddot\gamma(t)\times\dot\gamma(t) = k(\dot\sigma(s))^3\, b(s)$$

$$\therefore\ \|\ddot\gamma\times\dot\gamma\| = k\,(\dot\sigma(s))^3$$

Let s be the arc length function and σ be the reparametrization of γ by its arc length

Then $\gamma(t) = \sigma(s)$

$$\therefore\ \gamma'(t) = \dot\sigma(s)\,\frac{ds}{dt}$$

$$\therefore\ \gamma''(t) = \ddot\sigma(s)\left(\frac{ds}{dt}\right)^2 + \dot\sigma(s)\,\frac{d^2s}{dt^2}$$

$$\therefore\ \gamma''(t)\times\gamma'(t) = (\ddot\sigma(s)\times\dot\sigma(s))\left(\frac{ds}{dt}\right)^3 = (k\,\hat n\times\hat t)\left(\frac{ds}{dt}\right)^3$$

$$\therefore\ k(t) = \frac{\|\gamma'\times\gamma''\|}{\|\gamma'\|^3}$$

$$\tau = n'\cdot b = n'\cdot(t\times n)$$

Now $\gamma''(t) = k\,n\left(\frac{ds}{dt}\right)^2 + \hat t\,\frac{d^2s}{dt^2}$

$$\gamma'''(t) = \cancel{k'}\ \frac{dk}{dt}\,n\left(\frac{ds}{dt}\right)^2 + k\left(\frac{dn}{ds}\right)\left(\frac{ds}{dt}\right)^3 + \hat t\,\frac{d^3s}{dt^3} + k\,\hat n\,\frac{ds}{dt}\cdot\frac{d^2s}{dt^2}$$

$$\therefore\ \gamma'''(t)\times\hat t = \left(\frac{dk}{dt}\right)(-\hat b)\left(\frac{ds}{dt}\right)^2 + k\left(\frac{d\hat n}{ds}\right)\times\hat t\left(\frac{ds}{dt}\right)^3 + k(-\hat b)\,\frac{ds}{dt}\cdot\frac{d^2s}{dt^2}$$

$$\therefore\ (\gamma'''(t)\times\hat t)\cdot\hat n = k\left(\frac{d\hat n}{ds}\times\hat t\right)\cdot\hat n\left(\frac{ds}{dt}\right)^3$$

$$= k((-k\,t + \tau b)\times t)\cdot\hat n\left(\frac{ds}{dt}\right)^3$$

$$= k\tau\left(\frac{ds}{dt}\right)^3$$

$$\therefore\ \left(\gamma'''\times\frac{\gamma'}{\|\gamma'\|}\right)\cdot\hat n = k\tau\,\|\gamma'\|^3$$

Now $\gamma'' = k\,\hat n\left(\frac{ds}{dt}\right)^2 + \hat t\,\frac{d^2s}{dt^2}$

$$\therefore\ \left(\gamma'''\times\frac{\gamma'}{\|\gamma'\|}\right)\cdot\gamma''(t) = k^2\tau\,\|\gamma'\|^5$$

$$\therefore\ \gamma'''\cdot(\gamma'\times\gamma'') = k^2\tau\,\|\gamma'\|^6$$

$$\therefore\ \tau = \frac{(\gamma'\times\gamma'')\cdot\gamma'''}{\|\gamma'\times\gamma''\|^2}$$

(Note: Ex 7·11 on p93 appears wrong)

Ex Let us calculate the torsion of the twisted cubic $\gamma(t) = (t, t^2, t^3)$:

Ans: $3/(9t^4 + 9t^2 + 1)$

In particular, it is not a plane curve (though this can be seen by algebraically).

Surfaces in $\mathbb{R}^3$:

A surface S in $\mathbb{R}^3$ is a subset S with the property that for each $p \in S$, $\exists$ a neighborhood V_p of p in $\mathbb{R}^3$ and a function

$x: U \to V_p$ where U is a subset of $\mathbb{R}^2$

such that

(i) x is differentiable to all orders

(ii) x is a homeomorphism onto its image $x(U)$

i.e x is injective, continuous and

$x^{-1}: V_p \cap S \to U$ is continuous (i.e there exists a continuous function from $V_p \to U$ whose restriction to $V_p \cap S$ is ~~conti~~ x^{-1})

(iii) $x_u \times x_v \neq 0$ throughout U where (u, v) are the coordinates on U

In other words the derivative Dx has rank 2 throughout U.

Def: The function $x: U \to S$ described above is called a coordinate patch. ~~or~~ a

Ex: (Plane) Let $\hat{q} \neq 0$ be a unit vector $\Pi_{\hat{q}, a}$ be the plane containing a and having normal vector $\hat{q}$. Assume $q_3 \neq 0$

$$x(u, v) = (u, v, q_3^{-1}((a_1 - u) q_1 + (a_2 - v) q_2) + a_3)$$

maps $\mathbb{R}^2$ onto $\Pi_{\hat{q}, a}$ and ~~so~~ defines a coordinate patch.

(2) Consider $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,/\, x_1^2 + x_2^2 + x_3^2 = 1\}$

Define $G: (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$

$$G(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

Then G is a bijective map from $(0, 2\pi) \times (0, \pi)$ onto

$$S - \{(\sin\phi, 0, \cos\phi) \,/\, 0 \leq \phi \leq \pi\}.$$

and defines a coordinate patch

To show that G^{-1} is continuous, for $(x, y, z) \in S$, observe that $-1 < z < 1$ on S and so $\phi = \cos^{-1} z$ is continuous taking values in $(0, \pi)$.

Now let $\mathrm{Arg}: \mathbb{C} - (\text{pos. real axis}) \to (0, 2\pi)$ be a continuous branch of the argument function

Then $\frac{x}{\sqrt{1-z^2}} = \cos\theta$, $\frac{y}{\sqrt{1-z^2}} = \sin\theta$

and $x \neq 1$.

So $\theta = \mathrm{Arg}\left(\frac{x + iy}{\sqrt{1-z^2}}\right)$ and we have that G^{-1} is continuous.

A direct computation confirms that $G_\theta \times G_\phi \neq 0$. At points along the meridian $(\sin\phi, 0, \cos\phi)$ a different coordinate patch is needed.

(3) Again consider $S = \{(x_1, x_2, x_3) \,/\, x_1^2 + x_2^2 + x_3^2 = 1\}$ but this time we take the six coordinate patches each defined on $U = \{(u, v) \in \mathbb{R}^2 \,/\, u^2 + v^2 < 1\}$.

$$G_3^{\pm}(u, v) = (u, v, \pm\sqrt{1 - u^2 - v^2})$$

Images of $G_3^{\pm}$ cover S minus the equator.

To cover the equator we take 4 more patches

$$G_2^{\pm}(u, v) = (u, \pm\sqrt{1 - u^2 - v^2}, v)$$

which together with $G_1^{\pm}$ covers the sphere except the points

$(\pm 1, 0, 0)$. This will be covered by the patches

$$G_1^{\pm}(u,v) = (\pm\sqrt{1-u^2-v^2},\ u,\ v).$$

The inverse maps are easily written down For example

$$(G_3^{+})^{-1}(x_1, x_2, x_3) = (x_1, x_2)$$

Check that

$DG_i^{\pm}$ has rank 2 everywhere.

(4) Discuss the sphere again but construct two coordinate patches using stereographic projections.

(5) Coordinate path for the hyperboloid of one sheet $x_1^2 + x_2^2 - x_3^2 = 1$

$$G(u,v) = (\sqrt{1+v^2}\cos u,\ \sqrt{1+v^2}\sin u,\ v)$$

defined on $(0, 2\pi) \times \mathbb{R}$.
Check that DG has rank 2 and that this coordinate ~~patch~~ patch covers the entire surface except for one hyperbola.
Construct a second coordinate patch to cover this.

(6) Hyperboloid again! (Ruled surface)
Write $x_1^2 - x_3^2 = 1 - x_2^2$

$$(x_1 - x_3)(x_1 + x_3) = (1-x_2)(1+x_2)$$

Now consider the line of ~~intersect~~
L_α described as

$$L_\alpha: \quad x_1 - x_3 = (1-x_2)\alpha$$
$$x_1 + x_3 = (1+x_2)/\alpha \quad ; \alpha \in \mathbb{R},\ \alpha \neq 0.$$

L_α lies entirely on the surface
Likewise the lines M_β given by

$$M_\beta: \quad x_1 - x_3 = (1+x_2)\beta$$
$$x_1 + x_3 = (1-x_2)/\beta \quad ; \beta \in \mathbb{R} - (0).$$

lies entirely on the surface.
Thus the hyperboloid contains two families $\mathcal{F}_1, \mathcal{F}_2$ of lines such that
(a) Any two lines in one family are skew
(b) Any line of one family meets every line of the other family
(c) Through each point on the surface there passes exactly one member of each of the family. - except for certain exceptional points.

Ex: Verify (a), (b), (c) and determine the exceptional points.
Use (c) to write a coordinate ~~path~~ patch for the surface as a function of α and β.
Thus the hyperboloid of one sheet is a doubly ruled surface

(7) Discuss the surface $z = x^2 - y^2$
The function $G: \mathbb{R}^2 \to \mathbb{R}^3$ given by

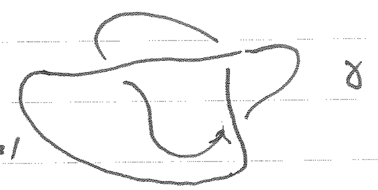
$$G(u,v) = (u, v, u^2 - v^2)$$

is a coordinate patch covering the entire surface.
But find another description as a doubly ruled surface.
Hint $(x+y)(x-y) = 1 \cdot z$.

(8) Helicoid:

The Cone: Let $\gamma(t)$ be a curve not passing through the origin.
The image of
$$G(s,t) = s\gamma(t)$$
is called a cone generated by γ. The cone will not in general be a surface as it may have self intersections.

Consider a γ described as in the figure. The cone determined by γ will in general not be a surface



but if we take a small arc of the curve γ and look at the cone generated by the arc it will be a surface (with one coordinate patch) and $G_s \times G_t$ is easily seen to be non zero.

Ex: Let γ be the curve of intersection of
$$y^2+z^2=1 \quad \text{and}$$
$$x^2+y^2=1$$
Consider the cone generated by γ and sketch the cone.

Use of the inverse function theorem: Suppose $\phi: \mathbb{R}^3 \to \mathbb{R}$ is a smooth map and c is a regular value of ϕ (~~vale~~ and $\phi^{-1}(c)$ is non empty (i.e $\forall\, p$ s.t $\phi(p)=c$, $\nabla\phi(p) \neq 0$) then Show that $S = \{ x \in \mathbb{R}^3 / \phi(x) = c \}$ is a surface.

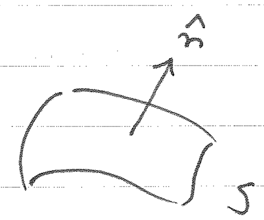
Theorem: Suppose S is a surface in $\mathbb{R}^3$ and $p \in S$. Let $x: U \to S$, $y: V \to S$ be two coordinate patches such that $p \in x(U) \cap y(V) = W$ then ~~[struck out]~~
Then
$$y^{-1} \circ x : x^{-1}(W) \to y^{-1}(W)$$
is differentiable, bijective and its inverse $x^{-1} \circ y : y^{-1}(W) \to x^{-1}(W)$ is also diff.

The proof follows immediately from the following

proposition: Let $x: U \to S$ be a coordinate patch. Then $x^{-1}: x(U) \to U$ is a smooth map in the sense that $\exists$ to each $p \in S$ a nbd Z of ~~$x(U)$~~ p and a smooth map $\phi: Z \to U$ whose restriction to $x(U)$ is x^{-1}.

proof: Since $x^{-1}: x(U) \to U$ is already bijective continuous, it makes sense to check smoothness locally. So let $p \in x(U)$ and $q = x^{-1}(p)$

Consider the map $F: U \times \mathbb{R} \to \mathbb{R}^3$
$$F(u,v,t) = x(u,v) + t\hat{n}$$



(which means the ~~level se~~ slice $t = \text{const } c$ maps to a surface S_c "parallel to S" by travelling a distance c along the normal vector $\hat{n}$);
where $\hat{n} = (x_u \times x_v) / \| x_u \times x_v \|$.

Let us calculate the Jacobian of F at $(q, 0)$:
~~$DF =$~~
$$F_u = x_u + t\frac{\partial \hat{n}}{\partial u}; \quad F_v = x_v + t\frac{\partial \hat{n}}{\partial v} \text{ and}$$
$$F_t = \hat{n} \text{ so that}$$

$$DF(q,0) = [x_u,\ x_v,\ \hat{n}(q)]$$

$\therefore$ Jacobian $F(q,0) = (x_u \times x_v)\cdot \hat{n} = \|x_u \times x_v\| \neq 0$.

By the inverse function theorem F is a local diffeomorphism between a nbd ~~of~~ W_1 $(q,0)$ and ~~$p \in S_1$~~ W_2 of $p \in S \subset \mathbb{R}^3$

Let $G = F^{-1}$ which is a smooth map from the nbd W_2 of p to W_1

Now
$$G(x(u,v)) = G(x(u,v) + t\hat{n})\big|_{t=0}$$
$$= G(F(u,v,t))\big|_{t=0}$$
$$= (u,v,0)$$

Thus $x^{-1}\big|_{W_2 \cap S} = (\pi \circ G)\big|_{W_2 \cap S}$

(π is the projection map
$(u,v,\lambda) \longmapsto (u,v)$
$W_1 \to U$)

i.e $x^{-1}\big|_{W_2 \cap S}$ is the restriction to $W_2 \cap S$ of the smooth map $\pi \circ G$ defined on a nbd of p.

Differentiability:

Let S_1, S_2 be two surfaces in $\mathbb{R}^3$ and $F: S_1 \to S_2$ be a continuous map. We now define the notion of smoothness of F by employing local coordinate patches.

Let $p \in S_1$ and $q = F(p)$.

Choose a coordinate patch (V,y) at $q \in S_2$ viz $y: V \to S \subset \mathbb{R}^3$.

By continuity $\exists$ a neighborhood W of p such that F maps $W \cap S_1$ into $y(V)$.

Now consider a coordinate patch (U,x) at p

~~Then, $y^{-1} \circ F \circ x|$~~

$$\begin{tikzcd} S_1 \arrow[r, "F"] & S_2 \\ U \arrow[u, leftrightarrow, "x"] \arrow[r, dashed] & V \arrow[u, "y"'] \end{tikzcd}$$
$y^{-1}\circ F \circ x$

Now x maps a nbd N of $x^{-1}(p)$ into W so that the composite function $y^{-1} \circ F \circ x\big|_N$ makes perfect sense and is a map from $N \to V$, N, V being open subsets of $\mathbb{R}^2$.

F is said to be differentiable at p if $y^{-1} \circ F \circ x\big|_N$ is differentiable at $x^{-1}(p) \in N$.

The notion is independent of the choice of coordinate patches (U,x) and (V,y), as is easily seen by using the theorem stated in the previous section.

Def: Surfaces S_1 and S_2 are said to be diffeomorphic if $\exists$ a homeomorphism $F: S_1 \to S_2$ such that F and F^{-1} are both differentiable in the sense described above.

Ex: (i) The sphere $x^2+y^2+z^2=1$ is diffeomorphic to the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Show that the relation of diffeomorphism is an equivalence relation.

(ii) The paraboloid $z = x^2+y^2$ and the part of the hyperboloid

$$1+z^2 = x^2-y^2 \quad ; \quad x \geq 1$$

are diffeomorphic.
Show that they are both diffeomorphic to the plane

(iii) Explain why the cylinder $x^2+y^2=1$ and the hyperboloid $x^2+y^2-z^2=1$ are diffeomorphic.

The tangent plane
Let p be a point on a surface S and (U, x) be a coordinate patch such that $p \in x(U)$. Let $x(u_0, v_0) = p$
Then the curves

$$u \mapsto x(u, v_0)$$
$$v \mapsto x(u_0, v)$$

are curves lying on S and their tangent vectors are

$$x_u(u_0, v_0) \text{ and } x_v(u_0, v_0).$$

Thus $\dfrac{x_u \times x_v}{\| x_u \times x_v\|}$ at (u_0, v_0) gives the normal vector to S

It is clear that if (V, y) is another coordinate patch containing p then

$$x_u = y_{\bar u} \frac{\partial \bar u}{\partial u} + y_{\bar v} \frac{\partial \bar v}{\partial u} ; \quad x_v = y_{\bar u} \frac{\partial \bar u}{\partial v} + y_{\bar v} \frac{\partial \bar v}{\partial v}$$

$$\therefore x_u \times x_v = (y_{\bar u} \times y_{\bar v}) \cdot \frac{\partial(\bar u, \bar v)}{\partial(u, v)} = (y_{\bar u} \times y_{\bar v})\, J(y^{-1} \circ x).$$

$$\therefore \frac{x_u \times x_v}{\| x_u \times x_v\|} = \pm \frac{y_{\bar u} \times y_{\bar v}}{\| y_{\bar u} \times y_{\bar v}\|}$$

Def: A surface S in $\mathbb{R}^3$ is said to be orientable if $\exists$ a collection of charts (U_α, x_α) such that $\{x_\alpha(U_\alpha)\}$ covers S
and $x_\beta^{-1} \circ x_\alpha$ has positive Jacobian on the overlap $U_\alpha \cap U_\beta$.
Let us call
Such a family of charts a coherent atlas.

If a coherent atlas exists (i.e if the surface is orientable) then computing the unit normal using a coherent atlas produces a <u>continuous</u> normal field

$$G : p \mapsto N(p) := N \circ x^{-1}(p)$$
$$S \mapsto S^2 \quad (\text{unit sphere in } \mathbb{R}^3)$$

called the Gauss map
Lemma:
~~Mean~~ Suppose that γ is a curve ~~also~~ on S and N_1, N_2 are two continuous normal fields along $\gamma : [0,1] \to S$.
then $N_1 = N_2$ at $\gamma(0)$
$\Rightarrow N_1 = N_2$ at $\gamma(1)$

<u>proof:</u> Clearly $N_1 = \pm N_2$ along γ
So consider the ~~continuous~~ Cont. map
$\phi : t \mapsto N_1 \cdot N_2$ which takes values ± 1
$\phi(0) = 1$
$\therefore \phi(1) = 1$ which means $N_1 = N_2$ at the terminal point as well.

Theorem: Suppose (U, x), (V, y) are two connected coordinate patches on S such that

(i) $U \cap V$ is disconnected

(ii) $J(x^{-1} \circ y) : U \cap V \to U \cap V$ changes sign then

S cannot be orientable.

pf. Suppose S is orientable, choose a continuous normal field

$N : S \to S^2$ along S

Let p, q be points in $U \cap V$ at which $J(x^{-1} \circ y)$ has diff. signs

and γ be a curve in U connecting p & q

σ be a curve in V connecting p and q

Also assume $J(x^{-1} \circ y) > 0$ at p and

$J(x^{-1} \circ y) < 0$ at q

Let $N' = \dfrac{x_u \times x_v}{\| x_u \times x_v \|}$; $N'' = \dfrac{y_{\bar{u}} \times y_{\bar{v}}}{\| y_{\bar{u}} \times y_{\bar{v}} \|}$

Thus $N' = N''$ at p.

$N' = -N''$ at q

Further we may assume N chosen such that

$N' = N'' = N$ at p

Then by the lemma, applied to γ we get

$N' = N$ at p

applied to σ we get $N'' = N$ at q

$\therefore$ $N' = N''$ at q which is a contradiction.

Use this fact to check that the Möbius band is not orientable.

(Ex 8.4 on p114 of the text).

The Gauss map would play a fundamental rôle in what follows.

Example: (i) Let us calculate the Gauss map for a point on the cylinder $x^2+y^2=1$

A coordinate patch is given by

$(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$; $\theta \in (0, 2\pi)$, $z \in \mathbb{R}$

$N(\theta, z) = (\cos\theta, \sin\theta, 0)$

Image of the Gauss map is the equator on S^2.

(ii) For $S = S^2$ the Gauss map is the identity map

(iii) For S a piece of the plane the Gauss map is constant.

(iv) For the paraboloid $z = x^2+y^2$,

$(x, y) \mapsto (x, y, x^2+y^2)$,

$N(x,y) = (-2x, -2y, 1) / \sqrt{1+4x^2+4y^2}$

Show that if $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a smooth map and c is a regular value of ϕ with $\phi^{-1}(c)$ non-empty then $\phi^{-1}(c)$ is an orientable surface.

First show that if S admits a continuous unit normal field then S is orientable.

The second fundamental form :

Let S be an orientable surface and $N: S \to \Sigma$ be the gauss map.
Then differentiating the equation,
$\langle N, N \rangle = 1$
we see that $\langle N_u, N \rangle = \langle N_v, N \rangle = 0$ so that N_u, N_v both belong to the tangent plane $T_p S$

Thus we can define a linear transformation $W: T_p S \to T_p S$ given by*

$$W(x_u) = N_u$$
$$W(x_v) = N_v$$

Now $\langle W(x_u), x_v \rangle$
$= \langle N_u, x_v \rangle$

But diff. the equations $\langle N, x_v \rangle = 0$ w.r.t u & $\langle N, x_u \rangle = 0$ w.r.t v we get
$\langle N_u, x_v \rangle = \langle N_v, x_u \rangle$

Hence $\langle W(x_u), x_v \rangle = \langle x_u, W(x_v) \rangle$
from which it follows at once that

$\langle W t_1, t_2 \rangle = \langle t_1, W t_2 \rangle$ for any pair of tangent vectors $t_1, t_2 \in T_p S$.

Def: The linear map W is called the Weingarten map.

* Very soon we shall see an invariant description, i.e one that does not refer to specific basis for $T_p S$.

Thm: The Weingarten map is self-adjoint with respect to the standard inner product on $T_p S$ (inherited from the ambient space $\mathbb{R}^3$)

Def. The quadratic form
$\mathrm{II}_p : T_p S \to T_p S$ given by
$t \mapsto \langle W(t), t \rangle$
is called the second fundamental form on the surface ~~of~~ S.

By employing Lagrange multipliers we see that

$$\lambda = \sup_{\|t\|=1} \mathrm{II}_p(t) \quad \text{and} \quad \inf_{\|t\|=1} \mathrm{II}_p(t) = \mu$$

are both eigen values of ~~II_p~~ W and that they are orthogonal (due to self adjointness) if ~~t~~ $\lambda \neq \mu$.
If $\lambda = \mu$ then $\mathrm{II}_p(t) = \lambda |t|^2$ and so ~~$W(t) = \lambda t$~~ $W(t) = \lambda \, id_{T_p S}$

We now look at the geometrical interpretation of the second fundamental form.
Recall that if $\sigma(s)$ is a unit speed curve $\vec{t} = \dot{\sigma}(s)$ is the tangent vector at $\sigma(s)$ and $\frac{d\vec{t}}{ds} = k\hat{n}$ where $k(s)$ is the curvature

Now, Let S be a surface and $\hat{N}_p$ be the unit normal vector to S at $p \in S$.
Consider the plane $\wedge_p$ passing through p, containing $\hat{N}_p$

(*) This gives an invariant description of the Weingarten Map.

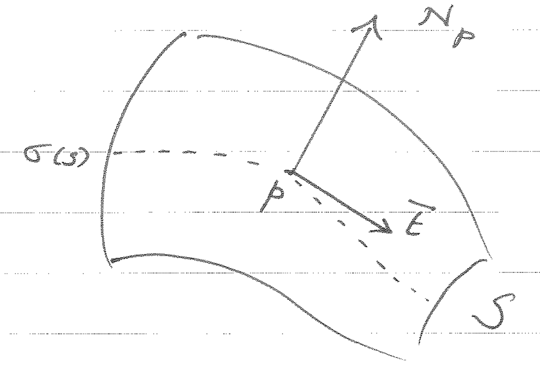
and a tangent vector $\vec{t}$ at p.

Choose a unit speed curve $\sigma : (-\epsilon, \epsilon) \to S$

such that $\sigma(0) = p$

$\dot\sigma(0) = \vec{t}$ and $\sigma(s) \in \Lambda$ for all s

Then $\hat{N}_p$ is ~~a~~ the normal to $\sigma(s)$ at p in the plane Λ_p



Note that $\sigma(s)$ is just the curve of intersection of Λ_p and S

Use a coordinate patch

$x : U \to S$ and let $t \mapsto (u(t), v(t))$

be the function $x^{-1} \circ \sigma$

then

$\dot\sigma(0) = \hat{t} = x_u \dot{u}(0) + x_v \dot{v}(0)$

(*) $\therefore \; W(\hat{t}) = N_u \dot{u}(0) + N_v \dot{v}(0)$

$$= \frac{d}{dt}\left((N \circ x) \circ (x^{-1} \circ \sigma)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}(N \circ \sigma)\Big|_{t=0}$$

~~Thus~~ But since $N \circ \sigma(t)$ is the unit normal to σ

$\frac{d}{ds}(N \circ \sigma(s)) = k \hat{t}$ by Frenet-Serret (σ is a plane curve so $\tau = 0$)

Where k is the curvature (upto sign) of σ at p.

$\therefore \; W(\vec{t}) = k_\sigma(p) \hat{t}$

and

$II_p(\hat{t}) = k_\sigma(p)$. Thus the second fundamental form gives the curvature of

Curves that are plane sections of S by the one parameter family of planes through p containing $\hat{N}_p$.

$k_\sigma(p)$ is more appropriately denoted by $k(p, \hat{t})$ which is the curvature at p in the direction $\hat{t} \in T_pS$.

Thus, the eigen values of the Weingarten map give the maximum and minimum curvature at p of all plane sections (at p) by planes passing through $\hat{N}_p$.

To calculate the Weingarten map, write

$N_u = \alpha x_u + \beta x_v$

$N_v = \gamma x_u + \delta x_v$

The matrix of W w.r.t the basis $\{x_u, x_v\}$ of T_pS is $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

Def: The eigen values of the Weingarten map are called the principal curvatures of S at p and the eigen vectors are the principal curvature directions.

Ex: Determine the principal curvatures and principal curvature directions at a point $(\cos\omega, \sin\omega, 0)$ on the hyperboloid

$x^2 + y^2 - z^2 = 1$.

Def: The Gaussian curvature of S at p is the determinant of the Weingarten map i.e the product of the principal curvatures.

The ~~trace~~ sum of the principal curvatures or the trace of the Weingarten map is called the mean curvature of S at p.

Let us compute the mean curvature of a surface which is the graph of a function $f : U \to \mathbb{R}$. There is only one coordinate patch

$$(u,v) \longmapsto (u, v, f(u,v))$$
$$x_u = (1, 0, f_u)$$
$$x_v = (0, 1, f_v)$$
$$N = (-f_u, -f_u, 1) / \sqrt{1+f_u^2+f_v^2}$$
Let $\Delta = (1+f_u^2+f_v^2)^{1/2}$
$$N_u = \frac{1}{\Delta}(-f_{uu}, -f_{uv}, 0) + \sigma_1 N$$
for some scalar function σ_1. Likewise
$$N_v = \frac{1}{\Delta}(-f_{uv}, -f_{vv}, 0) + \sigma_2 N$$
Thus writing
$$N_u = \alpha x_u + \beta x_v$$
$$N_v = \gamma x_u + \delta x_v$$
we get the equations

$$N_u \cdot x_u = -\frac{f_{uu}}{\Delta} = \alpha(1+f_u^2) + \beta f_u f_v$$
$$N_u \cdot x_v = -\frac{f_{uv}}{\Delta} = \alpha f_u f_v + \beta(1+f_v^2)$$

Solving,
$$\alpha = \frac{1}{\Delta^3}\left(-f_{uu}(1+f_v^2) + f_u f_v f_{uv}\right)$$
$$\beta = \frac{1}{\Delta^3}\left(-f_{uv}(1+f_u^2) + f_u f_v f_{uu}\right)$$

Similarly from $N_v \cdot x_u$ and $N_v \cdot x_v$ we get
$$\gamma = \left(-f_{uv}(1+f_v^2) + f_{vv} f_u f_v\right)/\Delta^3$$
$$\delta = \left(-f_{vv}(1+f_u^2) + f_u f_v f_{uv}\right)/\Delta^3$$

Thus
$$\begin{aligned} K = \frac{1}{\Delta^6}\Big\{ & f_{uu} f_{vv} (1+f_u^2)(1+f_v^2) + f_u^2 f_v^2 f_{uv}^2 \\ & - f_u f_v f_{uv}\left(f_{uu} + f_{vv} + f_{uu} f_v^2 + f_{vv} f_u^2\right) \\ & - f_{uv}^2 (1+f_u^2)(1+f_v^2) - f_{uu} f_{vv} f_u^2 f_v^2 \\ & + f_{uv}(1+f_u^2) f_{vv} f_u f_v \\ & + f_{uv}(1+f_v^2) f_{uu} f_u f_v \Big\} \end{aligned}$$

$$= \frac{1}{\Delta^6}\Big\{ f_{uu} f_{vv}(1+f_u^2)(1+f_v^2) + f_u^2 f_v^2 f_{uv}^2 - f_{uv}^2(1+f_u^2)(1+f_v^2) - f_{uu} f_{vv} f_u^2 f_v^2 \Big\}$$

$$= \frac{1}{\Delta^6}\Big\{ (f_{uu} f_{vv} - f_{uv}^2)(1+f_u^2)(1+f_v^2) - f_u^2 f_v^2 (f_{uu} f_{vv} - f_{uv}^2) \Big\}$$

$$= \frac{1}{\Delta^6} \cdot (f_{uu} f_{vv} - f_{uv}^2)\,\Delta^2$$

$$= \frac{1}{\Delta^4}(f_{uu} f_{vv} - f_{uv}^2)$$

Thus we get

Thm: The Gaussian curvature of a graph
$$(u,v) \longmapsto (u, v, f(u,v))$$
vanishes identically iff f satisfies the Monge-Ampere equation
$$f_{uu} f_{vv} - f_{uv}^2 = 0.$$

Ex: Verify that for the Cone

$$(x,y) \mapsto (x, y, \sqrt{x^2+y^2}) \; ; \quad 0 < x^2+y^2$$

the Gaussian Curvature vanishes.

Ex: Determine the Gaussian Curvature of the Cylinder, plane and unit sphere.

Show that the Gaussian Curvature of a paraboloid $z = x^2+y^2$ is positive.

Show that the ~~hyperb~~ hyperboloid
$x^2+y^2-z^2 = 1$ has negative Gaussian Curvature.

The tangent surface to a Curve:

This is the locus of lines tangent to a given Curve.

Let $\gamma : [a,b] \to \mathbb{R}^3$ be a unit speed curve. The tangent line to the curve at a point $\gamma(s)$ is given by

$$t \mapsto \gamma(s) + t\dot{\gamma}(s)$$

So the tangent surface is given by the coordinate patch

$$r : (s,t) \mapsto \gamma(s) + t\dot{\gamma}(s)$$

Now $\quad r_s = \dot{\gamma}(s) + t\ddot{\gamma}(s)$

$\quad r_t = \dot{\gamma}(s)$

$$\therefore \quad N = \frac{r_s \times r_t}{\| r_s \times r_t \|} = \frac{\ddot{\gamma}(s) \times \dot{\gamma}(s)}{\| \ddot{\gamma}(s) \times \dot{\gamma}(s) \|}$$

$\therefore \quad N_t = 0$ and so the Weingarten map is singular. The Gaussian Curvature is zero.

* At an umbilical point W has matrix $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ w.r.t any basis and so they can be found from the equations $\beta = \gamma = 0$ and $\alpha = \delta$.

Determine the surface which is the locus of the one parameter family of lines

$$y = tx - t^3$$
$$z = t^3 y - t^6$$

and find the Gaussian curvature of the surface.

Robert J.T. Bell p 315.

Def: A point $p \in S$ is said to be an umbilic if the Weingarten map has equal ~~and nonzero~~ eigen values at p

* Show that the origin is an umbilical point of the paraboloid

$$z = x^2+y^2$$

What about the surface $z = (x^2+y^2)^2$?

Determine the umbilical points on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Suppose that S is a smooth surface (i.e of class C^∞) and $p \in S$ is such that $\exists\, \epsilon > 0$ such that planes parallel to T_pS at a distance $< \epsilon$ from it meet S along a circle. Is p an umbilical point?

Sol: Let us fix up the origin at p and take the tangent plane T_pS as the x-y plane. Then the surface may be parametrized near p as

$$(u,v) \mapsto (u, v, f(u,v))$$

where f is a function of $\sqrt{u^2+v^2}$ alone.

$$f(0) = 0$$

Now $f(r) = ar + br^2 + cr^3 + \dots$ say.

But the fact that f is smooth forces $a = 0$

and $f(u,v) = b(u^2+v^2) + \cancel{d}\, c\,(u^2+v^2)^{3/2} + \ldots$

Now we may assume $b = 1$ (by rescaling u, v)

$$x_u = (1, 0, 2u) + (0, 0, \tfrac{3}{2} d (u^2+v^2)^{1/2} 2u) + \ldots$$

$$x_v = (0, 1, 2v) + (0, 0, \tfrac{3}{2} d (u^2+v^2)^{1/2} 2v) + \ldots$$

At p

$$x_u = (1, 0, 0)$$

$$x_v = (0, 1, 0)$$

$$N = ((1, 0, 2u) + r^2 o(r)) \times ((0, 1, 2v) + r^2 o(r))$$

($r\,o(r)$ terms have vanishing derivatives at p)

$$\left.\begin{aligned} N_u &= (0, 0, 2) \times (0, 1, 0) \\ N_v &= (1, 0, 0) \times (0, 0, 2) \end{aligned}\right\} \text{ at } p$$

$$\therefore \quad N_u \cdot x_u = -2$$

$$N_u \cdot x_v = 0$$

$$N_v \cdot x_u = 0$$

$$N_v \cdot x_v = -2$$

So W is represented by the diagonal matrix $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ at p which means p is an umbilic.

Circular Sections

~~Umbilics~~ of the Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$$(a > b > c > 0)$$

The xy plane slices it along the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with minor axis along $(0, b, 0)$

Whereas the yz plane slices along an ellipse with major axis at $(0, \pm b, 0)$.

By rotating the planes through the y-axis we get a family of ellipses with one axis along the y-axis, ~~and~~ continuously decreasing from a major axis in one case and a minor axis in another.

For one of these planes the two axes of the ellipse must become equal.

Let $\lambda x + \mu z = 0$ be a general planes through the y-axis. It cuts the ellipsoid along an ellipse with axis $(0, b, 0)$. The other axis must be orthogonal to this and so we look for it in the form $(x_0, 0, z_0)$

$$\lambda x_0 + \mu z_0 = 0 \qquad (\lambda > 0)$$

$$\frac{x_0^2}{a^2} + \frac{z_0^2}{c^2} = 1$$

$$\therefore \quad z_0^2 = \lambda^2 \left(\frac{\mu^2}{a^2} + \frac{\lambda^2}{c^2} \right)^{-1}$$

$$x_0^2 = \mu^2 \left(\frac{\mu^2}{a^2} + \frac{\lambda^2}{c^2} \right)^{-1}$$

The ellipse as axes b and $\sqrt{\lambda^2+\mu^2} \left(\dfrac{\mu^2}{a^2} + \dfrac{\lambda^2}{c^2} \right)^{-\frac{1}{2}}$

So this will be a circular section if

$$\lambda^2 + \mu^2 = b^2 \left(\frac{\mu^2}{a^2} + \frac{\lambda^2}{c^2} \right)$$

i.e $\quad \lambda^2 \left(1 - \dfrac{b^2}{c^2} \right) = \mu^2 \left(\dfrac{b^2}{a^2} - 1 \right)$

i.e $\quad \lambda^2 (b^2 - c^2) = \mu^2 (a^2 - b^2)$

$$\pm \lambda \sqrt{b^2 - c^2} = \mu (a^2 - b^2)^{1/2}$$

Taking $\lambda = 1$ (WLOG), $\quad \mu = \pm \dfrac{\sqrt{b^2 - c^2}}{\sqrt{a^2 - b^2}}$

So we summarize,

The family of planes $x_0 \pm \dfrac{\sqrt{b^2-c^2}}{\sqrt{a^2-b^2}}\, z = k \quad (*)$

~~meet~~ slice the ellipsoid along circles which diminish to points when these planes touch the ellipsoid.

Let the point of contact be (x_1, y_1, z_1)

Then ~~we see~~ Comparing $(*)$ with

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} + \frac{z z_1}{c^2} = 1 \quad \text{we get}$$

$$y_1 = 0 \; ; \quad k = \frac{a^2}{x_1} \, , \quad \frac{a^2}{c^2} \frac{z_1}{x_1} = \pm \sqrt{\frac{b^2 - c^2}{a^2 - b^2}}$$

$$\text{and} \quad \frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} = 1$$

Which give the four umbilical points

$$(x_1, 0, z_1) \; ;$$

$$x_1^2 = \frac{a^4 (a^2 - b^2)}{(a^2 - c^2)(a^2 + c^2 - b^2)}$$

$$z_1^2 = \frac{c^4 (b^2 - c^2)}{(a^2 - c^2)(a^2 + c^2 - b^2)}$$

Further Exercises.

(1) Regard N as a map from S to $\mathbb{R}^3$ so that $N \circ x : U \to \mathbb{R}^3$ and $D(N \circ x)$ is a linear map

Let p be a point on S and $\hat{t}$ be a tangent vector to S at p and $\sigma : \mathbb{R} \to S$ be a Curve such that

$$\sigma(0) = p$$
$$\dot{\sigma}(0) = \hat{t}$$

Show that

$$W(\hat{t}) = \frac{d}{ds} (N \circ \sigma(s)) \Big|_{s=0}$$

$$= D(N \circ x) \frac{d}{ds} (x^{-1} \circ \sigma) \Big|_{s=0} .$$

$$= (DN) \, \hat{t} \quad \text{regarding } DN \text{ as a}$$

(2) Suppose $\sigma : \mathbb{R} \to S$ is a line of curvature on S i.e a unit speed curve on S such that $\dot{\sigma}(s)$ is a principal curvature direction at $\sigma(s)$.

Show that $\displaystyle \frac{d}{dt} (N \circ \sigma(s)) \Big|_{t=0} = \lambda \, \dot{\sigma}(s)$

λ is an eigen value of W i.e the associated principal curvature.

~~Thus $DN(\hat{t}) = \lambda \hat{t}$ for a~~

This formula is called Rodrigues formula.

(3) Suppose S_1, S_2 are two surfaces intersecting along a regular curve γ which is a line of curvature on S. Then the angle of intersection of S_1 and S_2 is constant along γ iff γ is also a line of curvature of S_2.

(4) Determine the lines of curvature on the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 .$$

(5) For a surface of revolution show that the lines of curvature are parallels and meridians

(6) Determine explicitly the entries $\alpha, \beta, \gamma, \delta$ of the Weingarten map w.r.t the basis $\{x_u, x_v\}$ and write the result purely in terms of x_u, x_v, x_{uu}, x_{uv} and x_{vv} abbreviating

$$\frac{x_u \times x_v}{\| x_u \times x_v \|} = N \quad \text{if need be.}$$

That is to say formulas ~~in~~ free of $N_u, N_v,$

These equations are called the Weingarten equations.

(7) Determine the second fundamental form of Enneper's surface

$$X(u,v) = \left(u - \frac{u^3}{3} + uv^2;\ v - \frac{v^3}{3} + vu^2;\ u^2 - v^2\right)$$

and show that its mean curvature is zero.

(8) Find the mean curvature of the graph $(u,v) \mapsto (u, v, f(u,v))$ and determine the PDE that f would have to satisfy in order that its mean curvature $= 0$.

(9) Let Λ be a plane that cuts T_pS at angle θ and γ be the intersection of Λ with S. Find the curvature of γ

(Meusnier's theorem)

<u>Appendix</u>: Eigen values of a real symmetric matrix.

Suppose V is a finite dimensional inner product space and $A: V \to V$ is self adjoint. Then

$\lambda_1 = \sup_{|v|=1} \langle Av, v\rangle$ is the largest eigen value of A

$\lambda_2 = \sup_{\substack{|w|=1 \\ \langle w, v_1\rangle = 0}} \langle Aw, w\rangle$ is the next largest eigen value of A where v_1 is the unit vector such that

$$\lambda_1 = \langle Av_1, v_1\rangle = \sup_{|v|=1} \langle Av, v\rangle$$

etc;

$\inf_{|v|=1} \langle Av, v\rangle$ is the smallest eigen value of A.

<u>proof</u>: ~~Fixing a basis~~ Fix up a basis and write $v = \Sigma(x_1, \ldots, x_n)$ w.r.t this basis.

$Q(x) = \langle Ax, x\rangle$ and let's use Lagrange multipliers: $\sup_{|x|=1} \langle Ax, x\rangle$ is attained at say v_1 on the unit sphere.

$$L_1(x, \lambda) = Q(x) - \lambda(|x|^2 - 1)$$

Then $\frac{\partial L_1}{\partial x_j} = 0$; $\frac{\partial L}{\partial \lambda} = 0$ at v_1

Thus $2Av_1 - 2\lambda v_1 = 0$; $|v_1| = 1$

Thus λ is an eigen value of A ~~(can~~ with v_1 as the corresp. eigen vector. Call this λ_1

$\lambda_1 = \langle Av_1, v_1\rangle = \langle \lambda_1 v_1, v_1\rangle$

i.e The eigen value $\lambda_1 = \sup_{|v|=1} \langle Av, v\rangle$

Now let $\lambda_2 = \sup_{\substack{|v|=1 \\ \langle v, v_1\rangle = 0}} \langle Av, v\rangle = \langle Av_2, v_2\rangle$ (say where $|v_2| = 1$ and. $\langle v_1, v_2\rangle = 0$)

We now setup the Lagrangian

$$L_2(x, \nu, \mu) = Q(x) - \nu(|x|^2 - 1) - \mu\langle x, v_1\rangle$$

and at v_2

$$\frac{\partial L_2}{\partial x_j} = 0;\quad \frac{\partial L_2}{\partial \nu} = 0;\quad \frac{\partial L_2}{\partial \mu} = 0$$

Thus $2Av_2 - 2\nu v_2 - \mu v_1 = 0$; $|v_2| = 1$; $\langle v_2, v_1\rangle = 0$

Taking dot product with v_2 we get

(i) $\langle Av_2, v_2\rangle = \nu$

Taking dot product with v_1 gives

$$\mu = 2\langle Av_2, v_1\rangle - 2\nu\langle v_2, v_1\rangle$$
$$= 2\langle v_2, Av_1\rangle = 2\lambda_1\langle v_2, v_1\rangle = 0.$$

So that

$$A v_2 = \nu v_2$$
$$|v_2| = 1$$
$$\langle v_2, v_1 \rangle = 0.$$

$\therefore$ v_2 is an eigen vector of A with eigen value $\nu = \sup_{\substack{|v|=1 \\ \langle v, v_1\rangle = 0}} \langle A v, v\rangle \le \lambda_1$

Call $\nu = \lambda_2$.

Proceed further and optimize $Q(x)$ subject to the constraints $|x|^2 = 1$

$$\langle x, v_1 \rangle = \langle x, v_2 \rangle = 0$$

and we get the next eigen value.

If $v_1, \dots, v_{\ell-1}$ have been found such that

$$\langle v_i, v_j \rangle = \delta_{ij}$$
$$A v_i = \lambda_i v_i \; ; \quad \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_{\ell-1}$$

~~then~~ $\lambda_j = \sup_{\substack{|v|=1 \\ \langle v, v_k\rangle = 0 \; (k = 1, 2, \dots, j-1)}} \langle A v, v \rangle$

Let $L(x, \lambda, \nu_1, \dots, \nu_\ell)$

$$= Q(x) - \lambda(|x|^2 - 1) - \sum_{j=1}^{\ell-1} \langle x, v_j \rangle$$

$$\frac{\partial L}{\partial x_j} = 0 \; ; \quad \frac{\partial L}{\partial \lambda} = 0 \; ; \quad \frac{\partial L}{\partial \nu_k} = 0 \quad (k = 1, 2, \dots, \ell)$$

gives

$$2Ax - 2\lambda x - \sum_{j=1}^{\ell-1} \nu_j v_j = 0 \; ; \quad |x| = 1 \; ; \quad \langle x, v_j \rangle = 0$$

Dotting with $v_1, \dots, v_{\ell-1}$ in succession gives

$$\nu_j = 0 \text{ for each } j$$

whereby $Ax = \lambda x \; ; \; |x| = 1$

so that the optimization problem

$\sup_{\substack{|x|=1 \\ \langle x, v_j\rangle = 0 \; ; \; j = 1, 2, \dots, \ell-1}} \langle A x, x \rangle$ has a solution at an unit eigen vector v_ℓ orthogonal to $v_1, \dots, v_{\ell-1}$

and the value of the supremum $= \lambda_\ell$. the next eigen value in the list.

Cor: ~~An~~ ~~max~~ A self adjoint linear map has an orthonormal basis of eigenvectors.

The first fundamental form:

Let S be a surface in $\mathbb{R}^3$
at a point $p \in S$, the tangent plane T_pS inherits an inner product from its ambient space ~~an which is called the first~~ denoted by I_p
Thus we have the smoothly varying positive def. symmetric innerproduct
$$p \mapsto I_p$$ known as the first fundamental form.

If (U, x) is a coordinate patch with $x(u_0, v_0) = p$ then the tangents to the coordinate curves
$$u \mapsto x(u, v_0)$$
$$v \mapsto x(u_0, v)$$
generate the vector space T_pS and so I_p is uniquely determined by
$$I_p(x_u, x_u) = x_u(u_0, v_0) \cdot x_u(u_0, v_0)$$
$$I_p(x_u, x_v) = x_u(u_0, v_0) \cdot x_v(u_0, v_0)$$
$$I_p(x_v, x_v) = x_v(u_0, v_0) \cdot x_v(u_0, v_0).$$
These are the coefficients of the bilinear form I_p w.r.t. basis $\{x_u, x_v\}$

The traditional notation for these coefficients is
$$g_{11}(u,v) = x_u \cdot x_u$$
$$g_{12}(u,v) = g_{21}(u,v) = x_u \cdot x_v$$
$$g_{22}(v,v) = x_v \cdot x_v$$

(g_{ij}) are known as the components of the metric tensor.

Recall that if γ is a curve lying on S then the arc length of γ from ~~to to~~ t_0 to t is given by
$$s(t) = \int_{t_0}^{t} \|\gamma'(\lambda)\| d\lambda$$
$$= \int_{t_0}^{t} I_p(\dot\gamma \cdot \dot\gamma)^{1/2} d\lambda$$
But if $\sigma = x^{-1} \circ \gamma$ (a curve in U)
$= (u(t), v(t))$ say then
$$\gamma'(t) = x_u \frac{du}{dt} + x_v \frac{dv}{dt}$$
$$\therefore \langle \gamma'(t), \gamma'(t) \rangle = g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\frac{du}{dt}\frac{dv}{dt} + g_{22}\left(\frac{dv}{dt}\right)^2$$
Now
$$\left(\frac{ds}{dt}\right)^2 = g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\frac{du}{dt}\frac{dv}{dt} + g_{22}\left(\frac{dv}{dt}\right)^2$$
or
$$ds^2 = g_{11}du^2 + 2g_{12}\,du\,dv + g_{22}\,dv^2$$
Thus the differential of arc length ~~along~~ is expressible in terms of the components of the metric tensor (which is why it is called the metric tensor)

Transformation law of g_{ij}:

Let (V, y) be another coordinate patch and $p \in x(U) \cap y(V)$
and for $(u,v) \in x^{-1}(x(U) \cap y(V))$
$x(u,v) = y(\tilde u, \tilde v)$ for some $(\tilde u, \tilde v) \in y^{-1}(x(U) \cap y(V))$
and by implicit function thm, $\tilde u, \tilde v$ are smooth functions of (u,v)
$$\therefore x_u = y_{\tilde u}\frac{\partial \tilde u}{\partial u} + y_{\tilde v}\frac{\partial \tilde v}{\partial u}$$
$$x_v = y_{\tilde u}\frac{\partial \tilde u}{\partial v} + y_{\tilde v}\frac{\partial \tilde v}{\partial v}$$

$$g_{11}(u,v) = \tilde{g}_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial u} + \tilde{g}_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial u} + \tilde{g}_{22}\left(\frac{\partial\tilde{v}}{\partial u}\right)^2$$

$$g_{12}(u,v) = \tilde{g}_{11}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v} + \tilde{g}_{12}\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} + \tilde{g}_{21}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u} + \tilde{g}_{22}\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}$$

$$g_{22}(u,v) = \tilde{g}_{11}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{u}}{\partial v} + \tilde{g}_{12}\frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial v} + \tilde{g}_{22}\frac{\partial\tilde{v}}{\partial v}\frac{\partial\tilde{v}}{\partial v}.$$

Changing Notations slightly (U, x) with coord. (u_1, u_2) etc;
Let us denote $g_{ij} = x_{u_i}\cdot x_{u_j}$

$$g_{i'j'} = x_{v_i}\cdot x_{v_j}.$$

Then the transformation laws assume the more familiar form

$$g_{ij} = \sum_{i',j'=1}^{2} g_{i'j'}\frac{\partial v_{i'}}{\partial u_i}\frac{\partial v_{j'}}{\partial u_j}$$

That is (g_{ij}) are the components of a covariant tensor of rank 2.

<u>Area of a piece of a surface:</u>

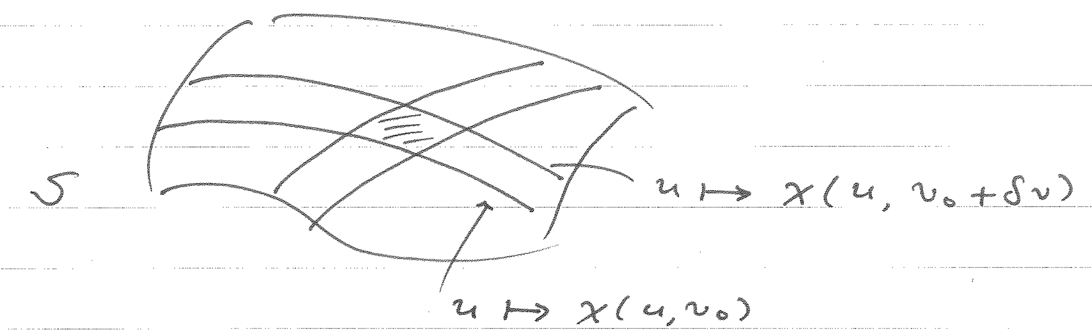
Let S be a surface and A be a piece of S contained in the image of a coordinate patch (x, U). The area of A is by definition

$$\iint_{x^{-1}(A)} \|x_u\times x_v\|\,du\,dv.$$

The motivation for this is of course

The fact that $\|x_u\times x_v\|$ is the first order approximation to the area of the curvilinear parallelogram determined by the coordinate curves

$$u\mapsto x(u,v_0) \;:\; u\mapsto x(u, v_0+\delta v)$$
$$v\mapsto x(u_0,v) \;:\; v\mapsto x(u_0+\delta u, v)$$



Now suppose that the region A is contained in $x(U)\cap y(V)$ where (U,x) and (V,y) are both coordinate patches.
Let $W_1 = x^{-1}(A)$; $W_2 = y^{-1}(A)$
The area of A may be computed in two ways namely

$$\iint_{W_1} \|x_u\times x_v\|\,du\,dv \quad \text{and}$$

$$\iint_{W_2} \|y_{\tilde{u}}\times y_{\tilde{v}}\|\,d\tilde{u}\,d\tilde{v}.$$

In order that the definition makes sense we must show that the displayed integrals produce the same value. Well, as observed earlier

$$x_u = y_{\tilde{u}}\frac{\partial\tilde{u}}{\partial u} + y_{\tilde{v}}\frac{\partial\tilde{v}}{\partial u}$$

$$x_v = y_{\tilde{u}}\frac{\partial\tilde{u}}{\partial v} + y_{\tilde{v}}\frac{\partial\tilde{v}}{\partial v}$$

$$\therefore \|x_u\times x_v\| = \|y_{\tilde{u}}\times y_{\tilde{v}}\|\left|\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\right|$$

$$\iint_{W_1} \| x_u \times x_v \| \, du\, dv = \iint_{W_1} \| y_{\tilde u} \times y_{\tilde v} \| \left| \frac{\partial(\tilde u, \tilde v)}{\partial(u,v)} \right| du\, dv$$

$$= \iint_{W_2} \| y_{\tilde u} \times y_{\tilde v} \| \, d\tilde u\, d\tilde v$$ by the ch. of. variables formula

How does one compute (or first define) area of a piece of S when this piece is not contained in the image of a single coordinate patch? Same issue for the arc length of a curve in terms of du, dv.

We shall merely sketch the salient features involved. First of all we shall only be concerned with $A \subset S$ such that A is compact and boundary of A is a finite union of piecewise smooth arcs (otherwise the Jordan content may not make sense and one may be forced to employ Lebesgue theory).

Now use the fact that every open cover has a ~~L~~ Lebesgue number and so $\mathbb{R}^3$ can be chopped into a family of non overlapping cubes each of whose ~~sides~~ sides is less than $1/3$ of the Lebesgue number*

Only finitely many of these cubes meet S and each slices S along a piece that lies entirely in one of the coordinate patches.

Now add up the contributions from each of these cubes.

* for the covering consisting of images of coordinate patches.

Exercises:

Determine the components of the first fundamental form of

(i) The paraboloid, $(u,v) \mapsto (u, v, u^2+v^2)$

(ii) The hyperbolic paraboloid $(u,v) \mapsto (u, v, u^2-v^2)$

(iii) The surface of revolution obtained by rotating $y = f(x)$ about the x-axis

(iv) For the hyperboloid of one sheet $x^2+y^2-z^2=1$ and the hyperbolic paraboloid $z = xy$ find the area of the portion of the surface contained within a quadrilateral determined by four rulings (two of each family).
The integral may not be computable explicitly

(v) Find the area of a piece of the tangent surface to a (unit speed) curve γ and also a piece of a cone determined by γ (not passing through the origin)

(vi) Consider the curve of intersection of
$$y^2+z^2=1 \quad \text{and} \quad x^2+z^2=1$$
and S be the cone determined by this curve.
Find the area of this cone.

Tangential Components of the acceleration vector or the Covariant derivative.

Now suppose that $\gamma : I \to S$ is a curve on S then $\dot\gamma(t)$ lies on the tangent space to S at $\gamma(t)$ but the acceleration vector $\gamma''(t)$ need not be tangential to S (even if γ is unit speed). For instance look at the "arctic Circle" on a sphere $x^2+y^2+z^2=1$ namely, the intersection of the sphere with the plane $z = c$, $c \neq 0$. The acceleration vector points towards the center of this circle $(0,0,c) \neq$ Origin and so is not tangent to the sphere.

Let us determine the components of the acceleration vector along the tangent plane. Known as the Covariant derivative of the velocity vector field.

Before taking this up it is useful to look at the case of Coordinate curves on a Coordinate patch (U, x) on the surface.

Let us Consider the Coordinate curves

$$u \mapsto x(u, v_0) : v \mapsto x(u_0, v)$$

through a point $x(u_0, v_0) \in S$

Differentiating the relations

$$g_{11} = x_u \cdot x_u ; \quad g_{12} = x_u \cdot x_v ; \quad g_{22} = x_v \cdot x_v$$

w.r.t u and v we obtain a system of six equations which determine

$$x_{uu} \cdot x_u, \quad x_{vv} \cdot x_u ; \quad x_{uv} \cdot x_u$$

$$x_{uu} \cdot x_v ; \quad x_{vv} \cdot x_v ; \quad x_{uv} \cdot x_v$$

Easily,

$$x_u \cdot x_{uu} = \frac{1}{2}\frac{\partial g_{11}}{\partial u}$$

$$x_v \cdot x_{vv} = \frac{1}{2}\frac{\partial g_{22}}{\partial v}$$

$$x_u \cdot x_{uv} = \frac{1}{2}\frac{\partial g_{11}}{\partial v}$$

$$x_v \cdot x_{uv} = \frac{1}{2}\frac{\partial g_{22}}{\partial u}$$

$\therefore$ ~~x_{uu}~~. Diff $\partial_u g_{12} = \partial_u (x_u \cdot x_v)$ we get

$$\frac{\partial g_{12}}{\partial u} = x_{uu} \cdot x_v + x_u \cdot x_{vu}$$

$$= x_{uu} \cdot x_v + \frac{1}{2}\frac{\partial g_{11}}{\partial v}$$

$$\therefore \quad x_{uu} \cdot x_v = \frac{\partial g_{12}}{\partial u} - \frac{1}{2}\frac{\partial g_{11}}{\partial v}.$$ Likewise

$$x_{vv} \cdot x_u = \frac{\partial g_{12}}{\partial v} - \frac{1}{2}\frac{\partial g_{22}}{\partial u}$$

We are now ready to calculate the projections of x_{uu}, x_{vv} and x_{uv} on $T_p S$.

Well,

$$x_{uu} = \lambda x_u + \mu x_v + \nu N$$

Taking dot product w.r.t x_u, x_v we get

$$x_{uu} \cdot x_u = \lambda g_{11} + \mu g_{12}$$

$$x_{uu} \cdot x_v = \lambda g_{12} + \mu g_{22}$$

Which can be immediately solved using Crammer's rule:

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \frac{1}{2}(g_{ij})^{-1} \begin{bmatrix} \partial_u g_{11} \\ 2\partial_u g_{12} - \partial_v g_{11} \end{bmatrix}$$

$$= \frac{1}{2}[g^{ij}] \begin{bmatrix} \partial_u g_{11} \\ \partial_u g_{12} + \partial_u g_{21} - \partial_v g_{11} \end{bmatrix}$$

where $[g_{ij}]^{-1} = [g^{ij}]$

$$\therefore \lambda = \frac{1}{2}\left\{ g^{11}\frac{\partial g_{11}}{\partial u} + g^{12}\left(\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{21}}{\partial u} - \frac{\partial g_{11}}{\partial v}\right)\right\}$$

$$\mu = \frac{1}{2}\left\{ g^{21}\frac{\partial g_{11}}{\partial u} + g^{22}\left(\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{21}}{\partial u} - \frac{\partial g_{11}}{\partial v}\right)\right\}$$

~~usual~~ λ and μ are denoted by Γ_{11}^1 and Γ_{11}^2 respectively

$$\therefore x_{uu} = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + \lambda N$$

Similarly we can compute the coefficients of x_u, x_v in the projection of x_{vv} and x_{uv} on T_pS

$$x_{vv} = \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v + \alpha N$$

$$x_{uv} = \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + \beta N$$

The coefficients Γ_{ij}^k are known as Christoffel symbols. The equality of mixed partials is expressed as

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell} g^{k\ell}\left(\frac{\partial g_{\ell i}}{\partial u_j} + \frac{\partial g_{\ell j}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_\ell}\right)$$

Exercise: Verify this. We have done it for $\Gamma_{11}^1, \Gamma_{11}^2$. Need to do it for the other four.

Suppose that $\gamma : ~~\mathbb{R}~~ I \to S$ is a curve ~~[illegible]~~ and $\sigma = x^{-1}\circ\gamma = (u(t), v(t))$

$$\gamma' = (x\circ\sigma) = x_u u' + x_v v'$$

$$\gamma'' = x_{uu}u'^2 + x_{vv}v'^2 + 2x_{uv}(u'v') + x_u u'' + x_v v''$$

$$= (\Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v)u'^2 + (\Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v)v'^2 + 2u'v'(\Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v)$$

$$+ x_u u'' + x_v v'' + \alpha N$$

for some scalar function α

Def: A curve on S whose acceleration has no tangential component is said to be a geodesic.

If now γ is a geodesic then

$$\left.\begin{aligned} \Gamma_{11}^1 u'^2 + \Gamma_{22}^1 v'^2 + 2u'v'\Gamma_{12}^1 + u'' = 0 \\ \Gamma_{11}^2 u'^2 + \Gamma_{22}^2 v'^2 + 2u'v'\Gamma_{12}^2 + v'' = 0 \end{aligned}\right\} (*)$$

which is a system of coupled second order ODES for u, v.

$x(u(t), v(t))$ is then the geodesic on S.

* Reads: $\sum \Gamma_{pq}^k u_p' u_q' + u_k'' = 0 \quad (k = 1, 2)$

Let us determine the geodesics on the unit sphere.

$$x(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

$$x_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$$

$$x_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$

$$g_{12} = x_\theta\cdot x_\phi = 0 ; \quad g_{11} = x_\theta\cdot x_\theta = \sin^2\phi$$

$$g_{22} = 1$$

Thus $g^{12}=0$; $g^{11}=\mathrm{cosec}^2\phi$; $g^{22}=1$

The Christoffel symbols are

$$\Gamma^1_{12} = \cot\phi; \quad \Gamma^2_{12}=0$$

$$\Gamma^1_{11} = 0; \quad \Gamma^2_{11} = -\sin\phi\cos\phi$$

$$\Gamma^1_{22} = 0; \quad \Gamma^2_{22} = 0$$

$$\therefore \quad \phi'' - \sin\phi\cos\phi\,\theta'^2 = 0$$

$$\theta'' + 2\cot\phi\,\theta'\phi' = 0.$$

~~$\theta'\sin^2\phi = 1$ (single A can be rescaled)~~

~~$\phi'^2 + \mathrm{cosec}^2\phi = \alpha$~~

~~(the other case is $\theta' = 0$)~~

Note that the spherical polar coordinates is well adapted to finding geodesics through the equator meeting it orthogonally but we proceed generally. The second equation integrates to

$$\theta'\sin^2\phi = \alpha \quad (\text{const of integration})$$

(or $\theta' = 0$)

The first equation now gives

$$\phi''\phi' - \sin\phi\cos\phi\cdot\frac{\alpha^2}{\sin^4\phi} = 0$$

$$\therefore \quad \frac{d}{dt}\left(\frac{1}{2}\phi'^2 + \frac{\alpha^2}{(+2)}\sin^{-2}\phi\right) = 0$$

$$\therefore \quad \phi'^2 + \alpha^2\,\mathrm{cosec}^2\phi = \beta$$

By rescaling time variable we may assume* $\beta = 1$

~~[illegible]~~

$$\therefore \quad \cos\phi = \sqrt{1-\alpha^2}\,\sin(t+t_0)$$

*This amounts to changing α by $\tilde{\alpha}$ throughout

and $$\theta' = \frac{\alpha}{\cos^2(t+t_0) + \alpha^2\sin(t+t_0)}$$

Let us assume that the geodesic crosses the equator at $t=0$ (so that $\phi = \pi/2$ when $t=0$)

Then $t_0 = 0$ and

$$\cos\phi = \sqrt{1-\alpha^2}\,\sin t$$

$$\theta' = \frac{\alpha}{\cos^2 t + \alpha^2\sin^2 t}$$

~~$\Rightarrow \tan(\theta+\lambda) = \alpha\cos\phi\tan t$~~

~~or $\sin\theta\cos\lambda + \cos\theta\sin\lambda = \alpha\cos\phi(\cos\theta\cos\lambda - \sin\theta\sin\lambda)$~~

~~which shows that the geodesic is the intersection curve of of the sphere and the plane surface~~

~~$y\cos\lambda + x\sin\lambda = \alpha z(x\cos\lambda - y\sin\lambda)$~~

$\therefore \quad \tan(\theta+\lambda) = \alpha\tan t$; λ is one more const. of integration.

Taking $\lambda = 0$,

$$\tan\theta = \alpha\tan t$$

$$\cos\phi = \sqrt{1-\alpha^2}\,\sin t$$

$$\therefore \quad \cos\theta = \frac{1}{\sqrt{1+\alpha^2\tan^2 t}}; \quad \sin\theta = \frac{\alpha\tan t}{\sqrt{1+\alpha^2\tan^2 t}}$$

$$\sin\phi = \sqrt{\cos^2 t + \alpha^2\sin^2 t} = \cos t\sqrt{1+\alpha^2\tan^2 t}$$

$$\therefore (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi) = (\cos t,\ \alpha\sin t,\ \sqrt{1-\alpha^2}\sin t)$$

This curve lies on the plane $\sqrt{1-\alpha^2}\,y - \alpha z = 0$

If $\lambda = 0$ is not assumed we would get a plane not passing through x-axis.

Note $\lambda = 0$ just means that the geodesic was at $(1,0,0)$ when $t=0$.

Note that a geodesic may not be reparametrizable for ~~its~~ the acceleration of the reparametrization would produce a tangential component.

<u>Thm</u>: (i) The geodesics have the property that the arc length function is proportional to t

(ii) The rescaling of time parameter keeps the geodesic equations invariant

<u>pf</u>: (ii) is immediate

(i) Let $\gamma = X(u(t), v(t))$ be a geodesic

$$\gamma'' = \sigma N \quad \text{for some scalar function } \sigma$$

$$\therefore \frac{d}{dt} \|\gamma'\|^2 = 2\langle \gamma', \gamma'' \rangle = 0$$

$\therefore \|\gamma'\|^2$ is constant in time $= c$ say

$$\therefore \quad s(t) = ct$$

Thus the geodesics may be assumed to be unit speed curves.

<u>The initial value problems of systems of ODEs</u>

Consider a system of second order ODEs:

$$x_i'' = F_i(t, x, x') ; \quad 1 \le i \le n \quad (*)$$

where F_i are smooth functions defined on an open set in $\mathbb{R}^{2n+1}$

For simplicity we assume that F_i are defined on a box

$$\Omega = [t_0 - 2a, t_0 + 2a] \times \{\|x - x_0\| < 2b\} \times \{\|y - y_0\| < 2b\}.$$

The initial value problem consists of a system $*$ together with a set of $2n$ initial conditions

$$x_i(t_0) = \alpha_i$$

$$x_i'(t_0) = \beta_i \qquad i = 1, 2, \dots, n$$

thereby providing a point $(t_0, \alpha, \beta) \in \Omega$

Assume that Ω is chosen such that $\alpha = x_0$, $\beta = y_0$

Define $y_i = x_i'$ $(i = 1, 2, \dots, n)$ and consider the system of first order ODEs

$$\left.\begin{aligned} x_i' &= y_i \\ y_i' &= F_i(t, x, y) \end{aligned}\right\} \quad i = 1, 2, \dots, n \qquad (**)$$

with initial conditions

$$x_i(t_0) = \alpha_i$$

$$y_i(t_0) = \beta_i$$

To each solution of $(*)$ with its initial conditions, there corresponds a unique solution of $(**)$ and vice versa.

So it suffices to discuss the system $(**)$ with the given initial conditions $x_i(t_0) = \alpha_i$, $y_i(t_0) = \beta_i$

Write $z = (x_1, y_1, \dots, x_n, y_n)$ and

$$\Omega = \{|t - t_0| < 2a\} \times \{\|z - z_0\| < 2b\}$$

$$\dot{z}_i = G_i(t, z_1, \dots, z_{2n})$$

$z_i(t_0) = c_i$ with $(t_0, c_1, \dots, c_{2n}) \in \Omega$ and G_i are smooth functions on Ω.

By Mean Value theorem

$$|G(t, \lambda) - G(t, \mu)| \le L \|\lambda - \mu\|$$

for $\lambda, \mu \in \mathbb{R}^{2n}$ such that

$$\|\lambda - z_0\| \le b \quad \text{and} \quad \|\mu - z_0\| \le b.$$

That is to say G satisfies a Lipschitz condition on a shrunken domain in z-space.

Let $M = \sup \{ \|G(t,z)\| \ / \ |t-t_0| \leq a, \ \|z-z_0\| \leq b \}$

and $h = \min\{a, \frac{b}{4M}\}$; $\tilde{\Omega} = \{|t-t_0| \leq h\} \times \{\|z-z_0\| \leq b\}$

Claim: For any point (t_0, ξ) with $\|\xi - z_0\| \leq b/3$

$\exists !$ solution $\phi(t)$ of the IVP defined on the interval $[t_0-h, t_0+h]$ with i.c ξ.

The solution depends continuously on ξ

Proof: (Existence) Solving the IVP is equivalent to solving the integral equation

$$z(t) = \xi + \int_{t_0}^{t} G(s, z(s))\,ds$$

Define the successive iterates

$$z_0(t) \equiv \xi \quad \text{and}$$

$$z_n(t) = \xi + \int_{t_0}^{t} G(s, z_{n-1}(s))\,ds. \quad (*)'$$

First of all we must ensure that the iterates are all defined on the common interval

$[t_0-h, t_0+h]$

In otherwords $(s, z_n(s)) \in \tilde{\Omega}$ so that $G(s, z_n(s))$ makes sense for each n and $s \in [t_0-h, t_0+h]$

Well, this is certainly true when $n=0$:

$$\|z_n(t) - \xi\| \leq \int_{t_0}^{t} \|G(s, z_{n-1}(s))\|\,ds$$

$$\leq |t-t_0|M$$

$$\leq hM \leq \frac{b}{4} < \frac{b}{3}$$

Now

$$\|z_n(t) - z_0\| \leq \|z_{n-1}(t) - z_0\| + \|z_0 - \xi\| \leq \frac{b}{3} + \frac{b}{3} = \frac{2}{3}b < b$$

So surely $(s, z_n(s)) \in \tilde{\Omega}$ if $(s, z_{n-1}(s)) \in \tilde{\Omega}$ for all $s \in [t_0-h, t_0+h]$. The claim is proved by induction.

Now we show $(z_n(t))$ converges uniformly on $[t_0-h, t_0+h]$. Well,

$$\|z_1(t) - \xi\| \leq \int_{t_0}^{t} \|G(s, z_0(s))\|\,ds$$

$$\leq M|t-t_0|$$

Assume inductively that

$$\|z_n(t) - z_{n-1}(t)\| \leq \frac{M|t-t_0|^n L^{n-1}}{n!}$$

Then $\|z_{n+1}(t) - z_n(t)\|$

$$\leq \int_{t_0}^{t} \|G(s, z_n(s)) - G(s, z_{n-1}(s))\|\,ds$$

$$\leq L \int_{t_0}^{t} \|z_n(s) - z_{n-1}(s)\|\,ds$$

$$\leq \frac{L^n M}{n!} \int_{t_0}^{t} |\delta - t_0|^n\,d\delta$$

$$\leq \frac{L^n M}{(n+1)!} |t-t_0|^{n+1}$$

Thus

$\sum_{n=0}^{\infty} \|z_{n+1}(t) - z_n(t)\|$ converges unif. on $[t_0-h, t_0+h]$

$\therefore \sum_{n=0}^{\infty} (z_{n+1}(t) - z_n(t))$ conv. unif. on $[t_0-h, t_0+h]$

The partial sums of the series conv. unif.

So $(z_n(t))$ conv. unif. on $[t_0-h, t_0+h]$ to $z(t)$ say.

Letting $n \to \infty$ in $(*)'$ we see that

$$z(t) = \xi + \int_{t_0}^{t} G(s, z(s))\,ds.$$ Now since the integrand in the RHS is continuous, $z(t)$ is C^1 and so

by the fundamental thm of Calculus

$$z'(t) = G(t, z(t))$$
$$z'(t_0) = \xi$$

Uniqueness and Continuity:

Changing the initial condition may change domain but we have ensured that for all ξ such that $\|\xi - z_0\| \le b/3$ the solutions are defined on $[t_0-h, t_0+h]$

To prove continuity w.r.t ξ as well as uniqueness we need the following

Lemma (Gronwall): Suppose $u, v : [a,b] \to \mathbb{R}$ are non negative functions such that for non neg constants C, k

$$u(t) \le C + k\int_{t_0}^{t} u(s)v(s)\,ds$$

then $u(t) \le C \exp \int_{t_0}^{t} k v(s)\,ds$

Proof: Assume first $C > 0$.

Put $U(t) = C + k\int_{t_0}^{t} u(s)v(s)\,ds \ge C > 0$ $(\because u, v \ge 0)$

Also by hypothesis $u(t) \le U(t)$

Now

$$\dot{U} = kuv \le Ukv$$

$$\therefore \log U - \log U(t_0) \le k\int_{t_0}^{t} v(s)\,ds \quad \left(\begin{array}{l}\because u \ge 0 \\ \therefore U \ge 0\end{array}\right)$$

$$\therefore U(t) \le U(t_0) \exp k\int_{t_0}^{t} v(s)\,ds \le C \exp \int_{t_0}^{t} k v(s)\,ds$$

Next, let $C = 0$.

Then surely $u(t) \le \frac{1}{n} + k\int_{t_0}^{t} u(s)v(s)\,ds$

by applying the previous case with $C = 1/n$ we infer

$$0 \le u(t) \le \frac{1}{n} \exp k\int_{t_0}^{t} v(s)\,ds$$

letting $n \to \infty$ we get

$$u(t) \equiv 0 = C \exp k\int_{t_0}^{t} v(s)\,ds$$

Now suppose $z(t)$ and $\tilde{z}(t)$ are two solutions of the IVP then

$$z(t) = \xi + \int_{t_0}^{t} F(s, z(s))\,ds$$

$$\tilde{z}(t) = \xi + \int_{t_0}^{t} F(s, \tilde{z}(s))\,ds$$

$$\therefore \|z(t) - \tilde{z}(s)\| \le L\int_{t_0}^{t} \|z(s) - \tilde{z}(s)\|\,ds$$

Applying Gronwall's lemma with $C = 0$, $k = L$, $v = 1$ and $u = \|z(s) - \tilde{z}(s)\|$ we conclude $u \equiv 0$ or $z(s) \equiv \tilde{z}(s)$.

Continuity w.r.t ξ

Again if $z(t, \xi)$ denotes the solution of the system of ODEs with initial value ξ and $W(t, \tilde{\xi})$ is the solution of the IVP with initial value $\tilde{\xi}$ then

$$z(t) = \xi + \int_{t_0}^{t} F(s, z(s))\,ds$$

$$W(t) = \tilde{\xi} + \int_{t_0}^{t} F(s, w(s))\,ds$$

$$\therefore \|z(t) - w(t)\| \le \|\xi - \tilde{\xi}\| + L\int_{t_0}^{t} \|z(s) - w(s)\|\,ds$$

Apply Gronwall with $C = \|\xi - \tilde{\xi}\|$ and $k = 1$, $v = 1$

$$\therefore \|z(t) - w(t)\| \le \|\xi - \tilde{\xi}\| \exp L\int_{t_0}^{t} ds$$

$$= \|\xi - \tilde{\xi}\| \exp L(t - t_0)$$

Proving Continuity w.r.t ξ.

It is true but harder to prove that the solution $\xi \mapsto z(t,\xi)$ is differentiable w.r.t ξ. However the following is important:

If we diff. the ODE
$$\dot z = F(t,z) \quad \text{w.r.t } \xi_j$$
$$\frac{\partial}{\partial \xi_j}\frac{dz}{dt} = \sum \frac{\partial F}{\partial z_k}\frac{\partial z_k}{\partial \xi_j}$$
$$\therefore \frac{d}{dt}\left(\frac{\partial z}{\partial \xi_j}\right) = \sum \frac{\partial F}{\partial z_k}\frac{\partial z_k}{\partial \xi_j} \quad \text{(Difficult step!)}$$
$$= DF(t, z(t,\xi))\frac{\partial z}{\partial \xi_j}$$

Thus each of the n-functions $\frac{\partial z}{\partial \xi_j}$; $j = 1, 2, \dots$ satisfies the same linear system of ODEs $\dot\xi = A(t)\xi$ where

$$A(t) = DF(t, z(t))$$

To obtain initial conditions observe that
$$\frac{\partial z}{\partial \xi_j}\bigg|_{t=0} = \frac{\partial}{\partial \xi_j}\left(z|_{t=t_0}\right) = \frac{\partial \xi}{\partial \xi_j} = \hat e_j$$

In particular
$$\frac{\partial(z_1, \dots, z_n)}{\partial(\xi_1, \dots, \xi_n)}\bigg|_{t=t_0} = 1.$$

By the Abel Liouville Thm,
$$W\left(\frac{\partial z}{\partial \xi_1}, \frac{\partial z}{\partial \xi_2}, \dots, \frac{\partial z}{\partial \xi_n}\right) = \exp \int_{t_0}^{t} (\text{Trace } A(s))\, ds$$
$$\therefore \frac{\partial(z_1, \dots, z_n)}{\partial(\xi_1, \dots, \xi_n)} = \exp \int_{t_0}^{t} (\text{Div } F)(s, z(s))\, ds$$

We shall not use this result though.

Returning now to geodesics,

Thm: Let S be a surface in $\mathbb{R}^3$ and $p \in S$. For each $w \in T_pS$, $\exists$ in a nbd U of p, a unique geodesic $\gamma : I \to S$ such that
$$\gamma(0) = p$$
$$\dot\gamma(0) = w$$

Well, choose a coordinate patch (U, x) containing p such that $x(u_0, v_0) = p$ $\therefore (u(t), v(t)) = x^{-1}\circ\gamma$
$$w = \dot\gamma(0) = x_u\,\dot u(0) + x_v\,\dot v(0)$$
Thus $\dot u(0)$, $\dot v(0)$ are expressible in terms of w and we have the initial conditions
$$\left.\begin{matrix} u(0) = u_0 \\ v(0) = v_0 \end{matrix}\right\},\ \dot u(0) \text{ and } \dot v(0)$$
With these we can uniquely solve the pair of ODEs

$$u'' + \Gamma_{11}^1 u'^2 + \Gamma_{22}^1 v'^2 + 2\Gamma_{12}^1 u'v' = 0$$
$$v'' + \Gamma_{11}^2 u'^2 + \Gamma_{22}^2 v'^2 + 2\Gamma_{12}^2 u'v' = 0$$
and $x(u(t), v(t))$ is the desired geodesic.

Remark: The geodesic through p in the direction $\frac{w}{\|w\|}$ is defined only on an interval I. In general I may not be whole of $\mathbb{R}$. The maximal interval on which γ exists is called the life span or time of existence of the geodesic. The finite life span is due to the non-linearity of the system of ODEs.

Thm: If S is a compact surface then

a11) The geodesics on S ~~have~~ have $\mathbb{R}$ as their interval of existence.

We shall not prove this theorem here.

<u>Geodesics</u> are <u>locally</u> <u>length</u> <u>minimizing</u> <u>Curves</u>:

Note that if p, q are two points on a surface S there may not be a geodesic on S connecting p and q

<u>Ex</u>: $S = \mathbb{R}^2 - (0)$ and $p = (1,1)$
$q = (-1,-1)$

There is no geodesic connecting $(1,1)$ and $(-1,-1)$.

Second, even if $\exists$ a geodesic connecting p and q, it need not be unique

<u>Ex</u>: Consider a sphere and two points on it. There are exactly two arcs of great circles joining them one of which is the line of shortest distance.

We prove the converse

<u>Thm</u>: If S is a surface and (U, x) is a coordinate patch and for $p, q \in x(U)$, $\gamma : [a,b] \to S$ is a curve ~~minimizing the~~ of least length joining p and q then γ is a geodesic.

<u>proof</u>: $\|\gamma'(t)\|^2 = \langle x_u \cdot u' + x_v \cdot v',\ x_u \cdot u' + x_v v' \rangle$
$= g_{11} u'^2 + g_{22} v'^2 + 2 g_{12} u' v'$

We change notations and write $u = u_1$,
$v = u_2$

* Well one ought to take $L(\gamma_\epsilon) = \int \|\gamma_\epsilon\| dt$ but with this $\frac{d}{d\epsilon} L(\gamma_\epsilon)\big|_{\epsilon=0}$ gives $\int \frac{d}{dt} \|\gamma_\epsilon\|^2 dt\Big|_{\epsilon=0} = 0$ since γ_0 may be assumed to have arc length parametrization.

$$\|\gamma'(t)\|^2 = \sum g_{ij} u_i' u_j'$$

Let $\phi_i : [a,b] \to \mathbb{R}$ be smooth curves such that $\phi_i = 0$ in a nbd. of a and b $(i = 1, 2)$

Then given $\epsilon > 0$ the curve
$\gamma_\epsilon = x(u_1 + \epsilon\phi_1,\ u_2 + \epsilon\phi_2)$ is a perturbation of γ and has the same endpts as γ

~~$\|\gamma_\epsilon'(t)\|^2 = \|x_{u_1}(u_1 + \epsilon\phi_1)(u_1' + \epsilon\phi_1') + x_{u_2}(u_2 + \epsilon\phi_2)(u_2' + \epsilon\phi_2')\|^2$~~

$= \cancel{g_{11}(u_1 + \epsilon\phi_1, u_2 +}$

$$\|\gamma_\epsilon'\|^2 = g_{11}(u + \epsilon\phi)(u_1' + \epsilon\phi_1')^2 + 2 g_{12}(u + \epsilon\phi)(u_1' + \epsilon\phi_1')(u_2' + \epsilon\phi_2') + g_{22}(u + \epsilon\phi)(u_2' + \epsilon\phi_2')^2$$

$$= \sum_{i,j} \Big(g_{ij}(u) + \sum_l \epsilon \frac{\partial g_{ij}}{\partial u_l}\phi_l + \epsilon^2 O(1)\Big)(u_i + \epsilon\phi_i)'(u_j + \epsilon\phi_j)'$$

$$= \sum_{i,j} \Big(g_{ij}(u) + \epsilon \sum \frac{\partial g_{ij}}{\partial u_l}\phi_l\Big)(u_i' u_j' + \epsilon u_i' \phi_j' + \epsilon u_j' \phi_i') + \epsilon^2 O(1)$$

$$= \|\gamma'\|^2 + \epsilon\Big\{\sum_{i,j}\sum_l \frac{\partial g_{ij}}{\partial u_l}\phi_l u_i' u_j' + \sum_{i,j} g_{ij} u_i' \phi_j' + \sum_{i,j} g_{ij} u_j' \phi_i'\Big\} + O(\epsilon^2)$$

* $L(\epsilon) = \int_a^b \|\gamma_\epsilon'(t)\|^2 dt$

Then $L'(0) = 0$ since γ minimizes arc length

$$\therefore \int_a^b \sum_{i,j} \sum_l \frac{\partial g_{ij}}{\partial u_l} \phi_l u_i' u_j' \, dt + 2 \int_a^b \sum_{i,j} g_{ij} u_j' \phi_i' \, dt = 0$$

We now integrate by parts.

$$\sum_l \int_a^b \phi_l \left(\sum_{i,j} \frac{\partial g_{ij}}{\partial u_l} u_i' u_j' \right) - 2 \int_a^b \sum_{i,j} \cancel{\frac{\partial}{\partial t}} (g_{ij} u_j')' \phi_i = 0$$

$$\sum_l \int_a^b \phi_l \left(\sum_{i,j} \frac{\partial g_{ij}}{\partial u_l} u_i' u_j' \right) - 2 \int_a^b \sum_{i,j} \left\{ \phi_i \left(g_{ij} u_j'' + u_j' \sum_k \frac{\partial g_{ij}}{\partial u_k} u_k' \right) \right\} dt = 0$$

$$\frac{1}{2} \sum_l \int_a^b \phi_l \left(\sum_{i,j} \frac{\partial g_{ij}}{\partial u_l} u_i' u_j' \right) dt - \int_a^b \sum_{i,j} \left\{ \phi_i u_j'' g_{ij} + \sum_k \frac{\partial g_{ij}}{\partial u_k} u_j' u_k' \phi_i \right\} dt = 0$$

In the second integral replace the dummy index i by l and use the fundamental ~~theorem~~ theorem of the Calculus of Variations.

Since ϕ_l are arbitrary we see that for each l,

$$\frac{1}{2} \sum_{i,j} \frac{\partial g_{ij}}{\partial u_l} u_i' u_j' - \sum_{\cancel{l},j} u_j'' g_{lj} - \sum_{\cancel{l},j} \sum_k \frac{\partial g_{lj}}{\partial u_k} u_j' u_k' = 0$$

$$\sum g_{lj} u_j'' + \sum_{j,k} \frac{\partial g_{lj}}{\partial u_k} u_j' u_k' - \frac{1}{2} \sum_{i,j} \frac{\partial g_{ij}}{\partial u_l} u_i' u_j' = 0$$

$$\therefore \sum g_{lj} u_j'' + \frac{1}{2} \sum_{j,k} \left(\frac{\partial g_{lj}}{\partial u_k} + \frac{\partial g_{lk}}{\partial u_j} \right) u_j' u_k' - \frac{1}{2} \sum_{i,j} \frac{\partial g_{ij}}{\partial u_l} u_i' u_j' = 0$$

Replacing k by i

$$\sum_j g_{lj} u_j'' + \frac{1}{2} \sum_{i,j} \left(\frac{\partial g_{lj}}{\partial u_i} + \frac{\partial g_{li}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l} \right) u_i' u_j' = 0$$

i.e. ~~$\sum_j g_{lj} u_j'' + \frac{1}{2} \sum_{i,j} \Gamma^l_{ij} u_i' u_j' = 0$. $(l \leq r, p)$~~

Mult by g^{pl} and sum over l

$$\sum_l \sum_j g^{pl} g_{lj} u_j'' + \frac{1}{2} \sum_{i,j} \sum_l g^{pl} \left(\frac{\partial g_{lj}}{\partial u_i} + \frac{\partial g_{li}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l} \right) u_i' u_j' = 0$$

$$\therefore \sum_j \delta^p_j u_j'' + \sum_{i,j} \Gamma^p_{ij} u_i' u_j' = 0$$

$$\therefore u_p'' + \sum \Gamma^p_{ij} u_i' u_j' = 0 \quad \text{for each } p.$$

Thus γ is a geodesic.

No.

Date

The exponential Map:

Let S be a surface and $p \in S$. By the fundamental existence uniqueness theorem on ODEs given any $v \in T_pS$, $\exists!$ geodesic

$\gamma_{p,v}(t)$ passing through p with tangent v at p. This is defined on an interval $(-2h, 2h)$. ~~Also $\exists$ a neighborhood W of p such that for all $q \in W$ and $v \in$~~

Let us now consider the set of all geodesics given by the initial conditions

$$u(0) = p$$
$$v(0) = p$$
$$u'(0) = \alpha \;;\quad v'(0) = \beta$$

Such that $\frac{d}{dt}\, x(u(t), v(t))\Big|_{t=0}$ has unit length.

i.e $\quad g_{11}(p)\alpha^2 + g_{22}(p)\beta^2 + 2g_{12}(p)\alpha\beta = 1$

Then the geodesic would be a unit speed geodesic.

Let then $\phi(t, 0, 0, \alpha, \beta)$ be the unique solution of the system of four first order ODEs for u, v.

~~Obs~~ These solutions are all defined on a common time interval $(-h, h)$

No.

Date

The paraboloid of revolution:

$$S: (u,v) \mapsto (u, v, u^2+v^2)$$
$$x_u = (1, 0, 2u)$$
$$x_v = (0, 1, 2v)$$
$$g_{11} = 1 + 4u^2; \quad g_{22} = 1 + 4v^2; \quad g_{12} = g_{21} = 4uv$$

$$g^{11} = \frac{1+4v^2}{\sqrt{1+4u^2+4v^2}}; \quad g^{22} = \frac{1+4u^2}{\sqrt{1+4u^2v^2}}$$

$$g^{12} = g^{21} = -4uv / \sqrt{1+4u^2+4v^2}$$

$$\Gamma^1_{11} = 4u(1 + \cancel{4uv}) / \sqrt{1+4u^2+4v^2}$$

$$\Gamma^1_{22} = 4u(\cancel{1+4u^2+4uv}) / \sqrt{1+4u^2+4v^2}$$

(Seems not to agree with Mritunjai's calculations!)

$$\Gamma^1_{12} = \Gamma^1_{21} = 0$$

$$\Gamma^2_{11} = 4v\,\cancel{(\cdots)} / \sqrt{1+4u^2+4v^2}$$

$$\Gamma^2_{22} = 4v / \sqrt{1+4u^2+4v^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = 0$$

The Geodesic Equations:

$$u'' + \frac{4u\,u'^2}{\sqrt{1+4u^2+4v^2}} + \frac{4u\,v'^2}{\sqrt{1+4u^2+4v^2}} = 0$$

$$v'' + \frac{4v\,u'^2}{\sqrt{1+4u^2+4v^2}} + \frac{4v\,v'^2}{\sqrt{1+4u^2+4v^2}} = 0.$$

Mult. first by v second by u and subtracting

$$u''v - uv'' = 0 \quad \text{or}$$

$$\frac{d}{dt}(u'v - uv') = 0 \quad \text{or}$$

for some constant c.

$$u'v - uv' = c$$

If $c=0$ then $\frac{u}{v}$ is constant and ~~As~~ are

~~these~~ rays in the uv plane. The geodesics are the meridians in this case.

Ex: In any surface of revolution the meridians are geodesics

Assume then $c \neq 0$. Write $u = r\cos\theta$
$v = r\sin\theta$

$$u' = r'\cos\theta - r\sin\theta\,\theta'$$
$$v' = r'\sin\theta + r\cos\theta\,\theta'$$
$$\therefore\ r^2\theta' = \text{constant} = c$$

Now

$$uu'' + vv'' + \frac{4r^2(u'^2+v'^2)}{\sqrt{1+4r^2}} = 0$$

$$\therefore\ (uu'+vv')' - (u'^2+v'^2) + \frac{4r^2(u'^2+v'^2)}{\sqrt{1+4r^2}} = 0$$

$$\therefore\ \frac{1}{2}(r^2)'' + (u'^2+v'^2)\left\{\frac{4r^2}{\sqrt{1+4r^2}} - 1\right\} = 0$$

$$\therefore\ \frac{1}{2}(r^2)'' + (r'^2 + r^2\theta'^2)\left\{\frac{4r^2}{\sqrt{1+4r^2}} - 1\right\} = 0.$$

$$rr'' + \frac{4r^2r'^2}{\sqrt{1+4r^2}} + \frac{4r^4\theta'^2}{\sqrt{1+4r^2}} - r^2\theta'^2 = 0.$$

$$\therefore\ rr'' + \frac{4r^2r'^2}{\sqrt{1+4r^2}} + \frac{4c^2}{\sqrt{1+4r^2}} - \frac{c^2}{r^2} = 0$$

This can be integrated and one can show that the geodesics (other than meridians) wind around densely

The exponential Map. (left incomplete).

Let $p \in S$ be a fixed point. By a simple compactness argument we see that $\exists\, h>0$ such that for every $w \in T_pS$ with $\|w\| = 1$,

There is a solution $\phi(t, p, w)$ of the geodesic equation with initial conditions (p, w) defined on $(-2h, 2h)$. That is to say as we vary w on the unit circle in T_pS, the geodesics have all a common domain $(-2h, 2h)$.

Thus if $w \in T_pS$ and $\|w\| \leq h$

$x \circ \phi(\|w\|, p, \frac{w}{\|w\|})$ makes sense and is a point of S.

Def: $E(w) = x \circ \phi(\|w\|, p, \frac{w}{\|w\|})$

$E: B \to S$ where B is the ball of radius h in T_pS is called the exponential map.

It is useful to have a slight reformulation

Let $w \in T_pS$ and $\|w\| \leq h$

Define $\psi(t) = \phi(\frac{t}{\|w\|}, p, w)$

Then ψ satisfies the DE for geodesics with initial conditions $x^{-1}(p, \frac{w}{\|w\|})$ same as the initial conditions of $\phi(t, p, \frac{w}{\|w\|})$

Thus, by uniqueness,

* Since $\phi(\frac{t}{\|w\|}, p, w) = \phi(t, p, \frac{w}{\|w\|})$ is defined for $|t| \leq \|w\|$

$\psi(t) = \phi(\frac{t}{\|w\|}, p, w) = \phi(t, p, \frac{w}{\|w\|})$

RHS is defined for $|t| < 2h$
LHS is defined for an interval around 0
which ~~(again by [illegible])~~ is certainly defined* for $|t| \leq \|w\|$

In particular $\psi(\|w\|) = \phi(1, p, w)$
$\phi(\|w\|, p, \frac{w}{\|w\|}) =$ ~~[illegible]~~

Thus the exponential map $E(w)$ is defined to be $\psi(1)$ where ψ is the unique solution of the geodesic equations with initial conditions (p, w)

This has two advantages:
(i) Makes sense even when $w = 0$
(ii) E is seen to be the time one map rather than evaluating at diff. times along diff. geodesics.

We shall now show that E is a diffeomorphism of a nbd of 0 onto a nbd of p on S.
Thus E sets up a coordinate chart on S which is intrinsic to S

The spherical polar coordinates on the sphere is an example. The meridians are geodesics and the parallels parametrize the family of geodesics.

~~We see that~~
~~$E(w) = \phi(1, p, w)$: $\|w\| \leq h$~~
~~We must now check that E is 1-1. If it were not then $\phi(1, p, w_1) = \phi(1, p, w_2)$ for some $w_1, w_2 \in T_pS$ which would mean two solutions arrive at the same point in a later time~~

So $\phi(\frac{t}{\|w\|}, p, w) = \phi(t, p, \frac{w}{\|w\|})$

and setting $t = \|w\|$ we get
$E(w) = x(\phi(1, p, w))$: $\|w\| \leq h$
We now want to compute the derivative of E w.r.t w to see that E has rank 2 everywhere on $\{\|w\| \leq h\}$

. Fixing a basis x_u, x_v for T_pS and writing
$w = a_3 x_u + a_4 x_v$
$E(w)$ ~~is~~ equals $x(u(t, \vec{a}), v(t, \vec{a}))$ where
$u(t, a_1, a_2, a_3, a_4)$, $v(t, a_1, a_2, a_3, a_4)$ are the solutions of the pair of ODEs

$$u_k'' + \sum_{i,j} \Gamma_{ij}^k u_i' u_j' = 0 \qquad (k = 1, 2)$$

with initial conditions $u_1(0) = u(0) = a_1$
$u_2(0) = v(0) = a_2$

$\dot{u}_1(0) = u'(0) = a_3$
$\dot{u}_2(0) = v'(0) = a_4$

a_1, a_2 are frozen as zero.

$$\frac{\partial E}{\partial a_3} = x_u \frac{\partial u}{\partial a_3} + x_v \frac{\partial v}{\partial a_3}$$

$$\frac{\partial E}{\partial a_4} = x_u \frac{\partial u}{\partial a_4} + x_v \frac{\partial v}{\partial a_4}$$

The ϕ in our previous discussion is then the pair (u, v)

$$\frac{\partial E}{\partial a_3} \times \frac{\partial E}{\partial a_4} = (x_u \times x_v)\left(\frac{\partial u}{\partial a_3}\frac{\partial v}{\partial a_4} - \frac{\partial u}{\partial a_4}\frac{\partial v}{\partial a_3}\right)$$

So it suffices to show that

$$\frac{\partial u}{\partial a_3} \cdot \frac{\partial v}{\partial a_4} - \frac{\partial u}{\partial a_4}\frac{\partial v}{\partial a_3} \neq 0. \text{ (at time } t=1)$$

Will not be Completed

The Compatibility conditions of Gauss, and Codazzi-Mainardi. Theorema Egregium.

We recall from the Chapter on Covariant differentiation, for a surface patch $x: U \to S \subset \mathbb{R}^3$

$$x_{uu} = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + \lambda_1 N$$
$$x_{uv} = \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + \lambda_2 N$$
$$x_{vv} = \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v + \lambda_3 N$$

We now apply the Conditions $\frac{\partial}{\partial v} x_{uu} = \frac{\partial}{\partial u} x_{uv}$

and $\frac{\partial}{\partial v} x_{uv} = \frac{\partial}{\partial u} x_{vv}$ and derive from it a

set of four equations known as the Gauss' equations. Among other things it would yield a formula for the Gaussian curvature in terms of the components of the metric tensor and their partial derivatives.

$$\frac{\partial}{\partial v} x_{uu} = \Gamma_{11}^1 x_{uv} + \Gamma_{11}^2 x_{vv} + \frac{\partial \Gamma_{11}^1}{\partial v} x_u + \frac{\partial \Gamma_{11}^2}{\partial v} x_v$$
$$+ \frac{\partial \lambda_1}{\partial v} N + \lambda_1 N_v$$
$$= \Gamma_{11}^1(\Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + \lambda_2 N) +$$
$$\Gamma_{11}^2(\Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v + \lambda_3 N) + \frac{\partial \Gamma_{11}^1}{\partial v} x_u + \frac{\partial \Gamma_{11}^2}{\partial v} x_v$$
$$+ \lambda_1 N_v + \frac{\partial \lambda_1}{\partial v} N$$

Recall that
$$N_u = \alpha x_u + \beta x_v$$
$$N_v = \gamma x_u + \delta x_v$$

and $\lambda_1 = x_{uu} \cdot N$

Now, $x_u \cdot N = 0 \Rightarrow x_{uu} \cdot N + x_u \cdot N_u = 0$

$$\therefore \lambda_1 = -x_u \cdot N_u = -\alpha g_{11} - \beta g_{12}$$

$$\frac{\partial}{\partial v} x_{uu} = x_u\Big(\Gamma_{11}^1\Gamma_{12}^1 + \Gamma_{11}^2\Gamma_{22}^1 + \frac{\partial \Gamma_{11}^1}{\partial v} - (\alpha g_{11}+\beta g_{12})\gamma\Big)$$
$$+ x_v\Big(\Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 + \frac{\partial\Gamma_{11}^2}{\partial v} - (\alpha g_{11}+\beta g_{12})\delta\Big) + \sigma_1 N$$

$$\frac{\partial}{\partial u} x_{uv} = \Gamma_{12}^1 x_{uu} + \Gamma_{12}^2 x_{vu} + \frac{\partial\Gamma_{12}^1}{\partial u}x_u + \frac{\partial \Gamma_{12}^2}{\partial u} x_v + \frac{\partial\lambda_2}{\partial u}N + \lambda_2 N_u$$

$$= \Gamma_{12}^1(\Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v) + \Gamma_{12}^2(\Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v)$$
$$+ (\Gamma_{12}^1 \lambda_1 N + \Gamma_{12}^2\lambda_2 N) + \frac{\partial\Gamma_{12}^1}{\partial u} x_u + \frac{\partial\Gamma_{12}^2}{\partial u} x_v$$
$$+ \lambda_2(\alpha x_u + \beta x_v) + \frac{\partial\lambda_2}{\partial u} N$$
$$= \cancel{x_u}\Big(\Gamma_{12}^1\Gamma_{11}^1 + \Gamma_{12}^2\Gamma_{12}^1 + \frac{\partial\Gamma_{12}^1}{\partial u} + \alpha\lambda_2\Big) x_u$$
$$+ \Big(\Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 + \frac{\partial\Gamma_{12}^2}{\partial u} + \beta\lambda_2\Big) x_v + \sigma_2 N$$

Comparing the coefficients of x_u and x_v gives the pair of equations:

$$\Gamma_{11}^2\Gamma_{22}^1 - \Gamma_{12}^1\Gamma_{12}^2 + \frac{\partial\Gamma_{11}^1}{\partial v} - \frac{\partial\Gamma_{12}^1}{\partial u} - \gamma(\alpha g_{11}+\beta g_{12}) - \alpha\lambda_2.$$

$$\Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2 + \frac{\partial\Gamma_{11}^2}{\partial v} - \frac{\partial\Gamma_{12}^2}{\partial u} - (\alpha g_{11}+\beta g_{12})\delta - \beta\lambda_2$$

Again, $\lambda_2 = x_{uv}\cdot N$

Diff $x_v \cdot N = 0$ w.r.t u gives

$$\lambda_2 = -x_v\cdot N_u = -x_v\cdot(\alpha x_u + \beta x_v) = -\alpha g_{12} - \beta g_{22}$$

Diff $x_u\cdot N=0$ w.r.t v gives

$$\lambda_2 = x_{uv}\cdot N = -x_u\cdot N_v = -x_u\cdot(\gamma x_u + \delta x_v) = -\gamma g_{11} - \delta g_{12}$$

it is convenient to use the second expression

$$\Gamma_{11}^2\Gamma_{22}^1 - \Gamma_{12}^1\Gamma_{12}^2 + \frac{\partial\Gamma_{11}^1}{\partial v} - \frac{\partial\Gamma_{12}^1}{\partial u} + g_{12}(\alpha\delta-\beta\gamma) = 0$$

Thus

$$g_{12}K = \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^1 + \frac{\partial\Gamma_{12}^1}{\partial u} - \frac{\partial\Gamma_{11}^1}{\partial v} \qquad (1)$$

and

$$g_{11}K = \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 + \frac{\partial\Gamma_{12}^2}{\partial u} - \frac{\partial\Gamma_{11}^2}{\partial v} \qquad (2)$$

Starting from $(x_{vv})_u = (x_{uv})_v$ we get the pair of equations (to be worked out. From McCleary)

$$g_{12}K = \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{22}^1\Gamma_{11}^2 + \frac{\partial\Gamma_{12}^2}{\partial v} - \frac{\partial\Gamma_{22}^2}{\partial u} \qquad (3)$$

$$g_{22}K = \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 - \Gamma_{12}^1\Gamma_{12}^1 - \Gamma_{12}^2\Gamma_{22}^1 + \frac{\partial\Gamma_{22}^1}{\partial u} - \frac{\partial\Gamma_{12}^1}{\partial v} \qquad (4)$$

Equations (1) - (4) are known as Gauss' eq^{n}s.